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How to correct the Bréguet range equation taking into account the fuel flow rate of the aircraft ?

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Abstract

This paper deals with the computation of the trimmed conditions in cruise as well as the prediction of the range taking into account the fuel flow rate, i.e. the weight decrease of the aircraft. A special emphasis was put on analytical computation, which leads to very simple expressions of correcting factors to apply on current fomulas used to compute the equilibrated conditions and range of the aircraft.
These analytical expressions have all been validated by a numerical simulation solving the flight dynamics equations with the introduction of a fuel flow rate proportional to the thrust.
It has been demonstrated that the Bréguet range formula used to estimate the performance in cruise at constant airspeed and angle of attack is optimistic and should be reduced by a factor k_e only depending on the equilibrated airspeed and Thrust Specific Fuel Consumption. This correction represents approx 0.6% of the range in the case of an airliner in cruise at a Mach number equal to 0.82.

1. Introduction

The prediction of the range and endurance of an aircraft is not a new subject. From the mid 20's onwards, an analytical formulation of the range and endurance for a propeller-driven airplane has been proposed with the so-called "Bréguet formulas",[2] even if the original computation should rather be put at the benefit of the Colonel Émile Dorand.[1] After World War II and the apparition of the jet engines, this formula has then been extended to the performance of airliners by considering that the fuel mass flow rate is proportional to the thrust in this case.
More recently,[6] a comparison between the actual range of aircraft collected by the US Department of Transportation and the estimation given by the Bréguet range equation has shown a good accuracy of this formula even if a residual difference (around 10%) remains for long-haul flights.
If a large part of the deviation between the prediction and the actual performance is due to the fraction of fuel burnt during the phases outside the steady state cruise (taxi, climb, descent…), another part is due to the assumptions made to perform the analytical computation (constant airspeed, L/D ratio, SFC…).
Amongst these is the assumption of a constant total weight of the aircraft for the computation of the aircraft trimmed conditions. If this makes sense for short term analyses (few minutes), for long term analyses (several hours), the weight decrease – due to the fuel flow rate – turns out in conflict with the assumption of a steady-level flight without any further input on the commands. Indeed, a simple analysis shows that the weight decrease automatically implies a continuous drift up with a residual flight path angle during a cruise at constant airspeed and angle of attack,[4] reason why such flight operation is also called the "cruise-climb" schedule.[9]
In light of this, the goal of this study was firstly to assess the effect of fuel flow rate on the aircraft trim states and secondly to correct the range computation with a special focus on the Bréguet range formula. In this article, a special emphasis was put on analytical computations leading to very simple expressions of the correcting factors which have very good accuracy with the numerical simulations.

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2. Assumptions and models

2.1 Atmosphere

In this article, the atmosphere state parameters are given by the International Standard Atmosphere (ISA) model.[3] For analytical computations, an exponential model of air density is used:

$$\rho = \rho_{\text{ref}}\, e^{a_h(h-h_{\text{ref}})} \tag{1}$$

where ρ_{ref} and h_{ref} denote reference air density and altitude depending on the atmosphere layer. This leads to:

$$\frac{\dot{\rho}}{\rho} = a_h \dot{h} \tag{2}$$

In the troposphere, i.e. from sea level to 11 km of altitude where the temperature decreases linearly, a_h is approximated by a least-squares curve fit of the actual evolution of the air density vs. altitude:

$$a_h = -1/9042\,\text{m}^{-1}$$

In the lower stratosphere, i.e. from 11 km to 25 km of altitude where the temperature is constant, a_h is directly computed with the ideal gas law:

$$a_h = -1.5777\;10^{-4}\text{m}^{-1}$$

2.2 Aerodynamic forces and moment

Drag and lift are the result of the projection of the aerodynamic force acting on the aircraft on the longitudinal aerodynamic axes:

$$\left\{ \begin{array}{lcl} D & = & \frac{1}{2}\rho S V^2 C_D \\ L & = & \frac{1}{2}\rho S V^2 C_L \end{array} \right. \tag{3}$$

In these expressions, V denotes the True Air Speed (TAS), S corresponds to the aircraft's reference area and the coefficients C_D and C_L are the drag and lift coefficient respectively. The drag coefficient C_D is expressed in terms of the lift coefficient and Mach number:[11]

$$C_D = C_D\,(C_L,\ \mathcal{M}) \tag{4}$$

Considering simple static models, the lift coefficient C_L is a function of the angle of attack α only :

$$C_L = C_L\,(\alpha)$$

Concerning the aerodynamic pitching moment M, we have:

$$M = \frac{1}{2}\rho S V^2 c\, Cm \tag{5}$$

The pitching moment coefficient Cm is expressed in terms of lift coefficient C_L, pitch rate q and elevator deflection δm:[5]

$$Cm = Cm\,(C_L, q, \delta m) \tag{6}$$

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2.3 Propulsion and fuel consumption

The thrust delivered to a jet engine is assumed to be a function of the altitude and TAS:[12]

$$T = k_T \rho V^{\lambda_T} \delta x \tag{7}$$

In this expression, the exponent λ_T is negative and depends on the by-pass ratio of the engine. The term δx denotes the throttle lever position, from 0 (idle) to 1 (full thrust).
For an aircraft equipped with jet engines, the fuel mass flow rate $\dot{m}_{fuel}$ is assumed to be proportional to the thrust:

$$\dot{m}_{fuel} = c_T T \tag{8}$$

where c_T denotes the thrust-specific fuel consumption (TSFC) which is assumed to be constant in this article.
And of course, the weight decrease of the aircraft $\dot{m}$ is the opposite of $\dot{m}_{fuel}$:

$$\dot{m} = -c_T T \tag{9}$$

2.4 Governing equations

Considering a symmetric aircraft with the thrust aligned with the fuselage reference line, the *longitudinal* equations of motion are the projections of the Newton's second law on the aerodynamic axes:[10]

$$\left\{ \begin{array}{lcl} \dot{V} & = & \frac{1}{m} \left(T \cos\alpha - D - mg \sin\gamma \right) \\ \dot{\gamma} & = & \frac{1}{mV} \left(T \sin\alpha + L - mg \cos\gamma \right) \end{array} \right. \tag{10}$$

In these expressions, α and γ denote the angle of attack (AoA) and flight path angle (FPA) respectively.
Assuming that the thrust is located at the aircraft center of gravity, the pitch rate comes from the aerodynamic pitching moment M only and we have in this case:

$$\dot{q} = \frac{M}{I_{YY}} \tag{11}$$

This set is completed by the kinematic equation in pitch:

$$\dot{\alpha} = q - \frac{1}{mV} \left(T \sin\alpha + L - mg \cos\gamma \right) \tag{12}$$

and altitude:

$$\dot{h} = V \sin\gamma \tag{13}$$

In these equations, the fuel flow rate $\dot{m}_{fuel}$ is already included into the thrust.[7,13] Moreover, the effect of the fuel mass flow rate on the pitching moment of inertia I_{YY} is assumed to be neglectible in this article.
In equilibrated flight conditions, the flight dynamics equations (Eq. (10)) directly give:

$$\left\{ \begin{array}{lcl} T_e \cos\alpha_e - D_e - mg \sin\gamma_e & = & 0 \\ T_e \sin\alpha_e + L_e - mg \cos\gamma_e & = & 0 \end{array} \right. \tag{14}$$

In these expressions, the subscript denote the *equilibrated conditions*.
Assuming neglectible AoA and low FPA, i.e. $\alpha_e \approx 0$ and $\gamma_e \ll 0$, this leads to the well-known equilibrated thrust and lift equations:

$$\left\{ \begin{array}{rcl} T_e - D_e & = & mg\gamma_e \\ L_e & = & mg \end{array} \right. \tag{15}$$

Moreover, considering a cruise at constant angle of attack α_e and TAS V_e, the range is predicted by the "Bréguet range equation" :

$$R = \frac{V_e f_e}{g c_T} \ln\left(\frac{m_i}{m_f} \right) \tag{16}$$

In this expression, f_e denotes the lift-over-drag ratio corresponding to the equilibrated lift coefficient C_{Le}, i.e. $f_e = \frac{C_{Le}}{C_{De}}$, whereas m_i and m_f correspond to the initial and final mass of the aircraft for the cruise phase.

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2.5 Case study

In this article, the aircraft is representative of a classic twin-engine liner from the Airbus wide-body family[8] with the following caracteristics :

- total mass $m = 130\,tons$
- reference area $S = 260\,m^2$
- thrust exponent $\lambda_T = -0.3$

For an *equilibrated* flight corresponding to a cruise performed at the altitude $h_e = 30\,000$ ft and Mach number $\mathcal{M} = 0.82$, the following correspondance between the equilibrated state parameters and commands of the aircraft was computed by solving the equilibrated flight equations (Eq. (15)):

$$\left\{\begin{array}{lcl} \delta m_e & = & -4.7° \\ \delta x_e & = & 0.91 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{lcl} V_e & = & 248.58\,\text{m/s} \\ \alpha_e & = & 0.4\,\text{deg} \\ \gamma_e & = & 0\,\text{deg} \\ q_e & = & 0 \end{array}\right. \tag{17}$$

Without any further input on the commands and at *a constant weight*, these parameters remain constant all along the flight.

3. Equilibrium analysis

3.1 Numerical results

From the conditions corresponding to the equilibrated flight (Eq. (17)), the evolution of the state vector of the aircraft vs. time was computed with the effect of the weight decrease due to the fuel flow rate. These results was obtained by numerically solving the longitudinal equations of motion (Par. (2.4)).
In this simulation, the fuel flow rate $\dot{m}_{fuel}$ given by (Eq. (8)) was introduced abruptly at the initial time. The figures (Fig. (1)) and (Fig. (2)) show that after a transitional phase (made of the phugoïd mode) and without any further input on the commands, the aircraft converges toward new equilibrated flight conditions different from the initial ones.

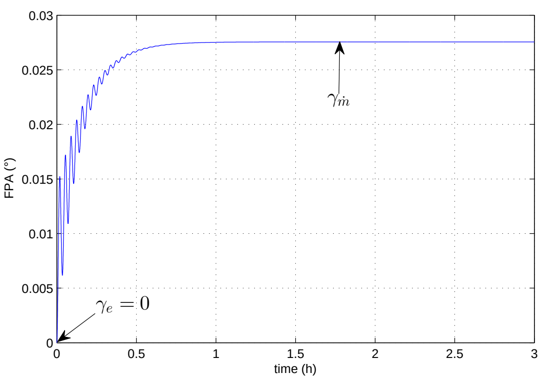

Figure 1: Evolution of the flight path angle vs. time.

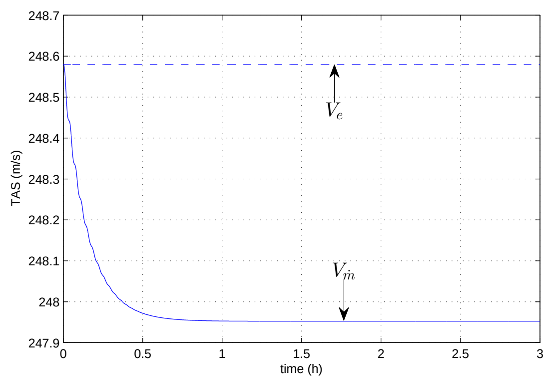

Figure 2: Evolution of the TAS vs. time.

Thus, when taking into account the weight decrease of the aircraft $\dot{m}$, the correspondance between the equilibrated state parameters and commands of the aircraft already computed at constant weight (Eq. (17)) is modified as follows:

$$\left\{\begin{array}{lcl} \delta m_e & = & -4.7° \\ \delta x_e & = & 0.91 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{lcl} V_{\dot{m}} & = & 247.9\,\text{m/s} \\ \alpha_{\dot{m}} & = & \alpha_e \\ \gamma_{\dot{m}} & = & 0.0276\,\text{deg} \\ q_{\dot{m}} & = & q_e \end{array}\right. \tag{18}$$

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In this expression and from now on, the subscript $\dot{m}$ denotes the equilibrated parameters taking into account the fuel flow rate of the aircraft.
It must be noted that the aircraft converges toward an equilibrated flight in a very light climb ($\gamma_{\dot{m}} > 0$) and a slightly lower speed ($V_{\dot{m}} < V_e$). Moreover, the longitudinal parameters α_e and q_e do not change as it's assumed that $\dot{m}$ does not produce any pitching moment (Eq. (11)).
In addition, the figure (Fig. (3)) shows that the equilibrated TAS $V_{\dot{m}}$ slightly increases when a correcting factor $\epsilon_{\delta x}$ is applied to the equilibrated thrust lever δx_e:

$$\delta x_e' = \delta x_e \left(1 + \epsilon_{\delta x}\right)$$

Then, an "optimal" correcting factor $\tilde{\epsilon}_{\delta x}$ given by the condition $V_{\dot{m}} = V_e$ can be computed with the use of the numerical computation. This also leads to an "optimal" thrust lever position $\widetilde{\delta x}_e$ given by:

$$\widetilde{\delta x}_e = \delta x_e \left(1 + \tilde{\epsilon}_{\delta x}\right) \tag{19}$$

The numerical simulation gives:

$$\tilde{\epsilon}_{\delta x} = 0.00632$$

The figure (Fig. (4)) confirms that the aircraft converges toward the same equilibrated TAS as its initial value in this case. In this figure, the phugoïd mode is still visible but its amplitude is very small in this case (maximum around 0.08 m/s). This oscillation is due to the fact that the FPA is initialized at 0 and the aircraft converges toward $\gamma_{\dot{m}}$ through the phugoïd mode. Thus, this transitional phase can be completely erased by initializing the FPA directly at $\gamma_{\dot{m}}$ in the simulation.

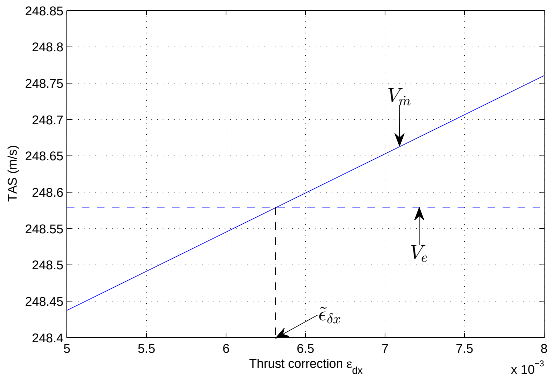

Figure 3: Evolution of the TAS vs. $\epsilon_{\delta x}$ and computation of the correcting factor $\tilde{\epsilon}_{\delta x}$.

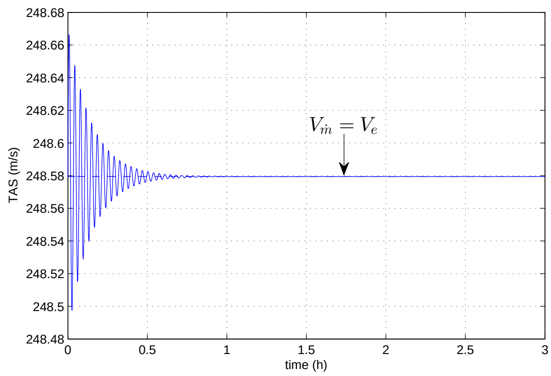

Figure 4: Evolution of the TAS vs. time with the correction $\tilde{\epsilon}_{\delta x}$.

With the correcting factor $\tilde{\epsilon}_{\delta x}$, the correspondance between the equilibrated state parameters and commands of the aircraft (Eq. (18)) is then shifted as follows:

$$\left\{ \begin{array}{rcl} \delta m_e & = & -4.7° \\ \widetilde{\delta x}_e & = & \delta x_e \left(1 + \tilde{\epsilon}_{\delta x}\right) \end{array} \right. \iff \left\{ \begin{array}{rcl} V_{\dot{m}} & = & V_e \\ \alpha_{\dot{m}} & = & \alpha_e \\ \gamma_{\dot{m}} & = & 0.0276° \\ q_{\dot{m}} & = & q_e \end{array} \right. \tag{20}$$

3.2 Analytical computation

Considering the equilibrated lift equation (Eq. (15)), as lift always equals weight, i.e. $L_{\dot{m}} = mg$, the decrease of weight due to the fuel flow rate implies at the same time a decrease of the lift. Moreover, the rate of decrease of the lift is given by:

$$\frac{\dot{L}_{\dot{m}}}{L_{\dot{m}}} = \frac{\dot{m}}{m}$$

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Assuming that $\dot{m}$ does not produce any pitching moment (Eq. (11)) and without any input on the pitch command, the lift coefficient C_L is constant which leads to:

$$\frac{\dot{\rho}_{\dot{m}}}{\rho_{\dot{m}}} = a_h \dot{h}_{\dot{m}} = \frac{\dot{m}}{m} \tag{21}$$

This expression clearly shows that the weight decrease due to the fuel flow rate, i.e. $\dot{m} = -\dot{m}_{fuel} < 0$, produces an increase of altitude given by $\dot{h}_{\dot{m}} = \frac{\dot{m}}{a_h m} > 0$.
Moreover, as $\dot{h}_{\dot{m}} = V_{\dot{m}} \sin \gamma_{\dot{m}}$, this leads to the following expression of the residual FPA $\gamma_{\dot{m}}$ coming from the weight decrease:

$$\sin \gamma_{\dot{m}} = \frac{\dot{m}}{a_h V_{\dot{m}} m}$$

Finally, assuming that $\gamma_{\dot{m}}$ is small and by introducing the expression of $\dot{m}$ (Eq. (9)), this can be written as:

$$\gamma_{\dot{m}} = \frac{-c_T g}{a_h f_e} \frac{1}{V_{\dot{m}}} \tag{22}$$

Additionaly, the thrust equation (Eq. (15)) written in equilibrated flight conditions taking into account the weight decrease $\dot{m}$ is given by:

$$T_{\dot{m}} - D_{\dot{m}} = mg\gamma_{\dot{m}} \tag{23}$$

By comparison with the equilibrated thrust equation at constant weight and $\gamma_e = 0$, i.e. $T_e = D_e = \dfrac{mg}{f_e}$, this leads to the relation:

$$\frac{T_{\dot{m}}}{T_e} - \frac{D_{\dot{m}}}{D_e} = f_e \gamma_{\dot{m}} \tag{24}$$

At a given altitude (ρ fixed), and considering the propulsion and drag models, we get:

$$\frac{T_{\dot{m}}}{T_e} = \left(\frac{V_{\dot{m}}}{V_e}\right)^{\lambda_T} \text{ and } \frac{D_{\dot{m}}}{D_e} = \left(\frac{V_{\dot{m}}}{V_e}\right)^2$$

The equation (Eq. (24)) then leads to :

$$\left(\frac{V_{\dot{m}}}{V_e}\right)^{\lambda_T} - \left(\frac{V_{\dot{m}}}{V_e}\right)^2 - k_e \left(\frac{V_{\dot{m}}}{V_e}\right)^{-1} = 0 \tag{25}$$

With:

$$\boxed{k_e = \frac{-c_T g}{a_h V_e}} \tag{26}$$

Assuming that the difference between $V_{\dot{m}}$ and V_e is small and $V_{\dot{m}} < V_e$, $V_{\dot{m}}$ is given by:

$$V_{\dot{m}} = V_e (1 - \epsilon_V) \tag{27}$$

Where the correcting factor ϵ_V is small ($\epsilon_V \ll 1$). Finally, a first order development of the equation (Eq. (25)) directly leads to an analytical expression of ϵ_V:

$$\epsilon_V = \frac{k_e}{2 - \lambda_T - k_e} \tag{28}$$

Considering (Eq. (22)), this gives for $\gamma_{\dot{m}}$:

$$\gamma_{\dot{m}} = \frac{k_e}{f_e} (1 + \epsilon_V)$$

Finally, when the optimal correcting factor $\tilde{\epsilon}_{\delta x}$ is applied to the thrust lever (Eq. (19)), the equilibrated thrust equation is given by:

$$\widetilde{T}_e - \widetilde{D}_e = mg\gamma_{\dot{m}} \tag{29}$$

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By comparison with the equilibrated thrust equation at constant weight and $\gamma_e = 0$, i.e. $T_e = D_e = \dfrac{mg}{f_e}$, this leads to the relation:

$$\frac{\widetilde{T}_e}{T_e} - \frac{\widetilde{D}_e}{D_e} = f_e \gamma_{\dot{m}} \tag{30}$$

At a given altitude (ρ fixed), and considering the propulsion and drag models, we have in this case:

$$\frac{\widetilde{T}_e}{T_e} = (1 + \tilde{\epsilon}_{\delta x}) \text{ and } \frac{\widetilde{D}_e}{D_e} = 1$$

This gives directly the expression of $\tilde{\epsilon}_{\delta x}$:

$$\tilde{\epsilon}_{\delta x} = f_e \gamma_{\dot{m}} = k_e (1 + \epsilon_V)$$

Considering that k_e and ϵ_V are small, these expressions can be simplified as follows:

$$\boxed{\left\{ \begin{array}{lcl} \gamma_{\dot{m}} & = & \dfrac{k_e}{f_e} \\ \epsilon_V & = & \dfrac{k_e}{2 - \lambda_T} \\ \tilde{\epsilon}_{\delta x} & = & k_e \end{array} \right.}$$

With these formulas, the numerical application gives:

$$\left\{ \begin{array}{lcl} \gamma_{\dot{m}} & = & 0.0273° \\ V_{\dot{m}} & = & 247.9 \text{ m/s} \\ \tilde{\epsilon}_{\delta x} & = & 0.00629 \end{array} \right.$$

Thus, the analytical expressions presented here are in very good accordance (difference < 1%) with the numerical results presented in the previous paragraph.

4. Performances analysis

4.1 Numerical results

The range has been computed between 2 moments, t_i and t_f during the equilibrated conditions presented in the previous section (Par. (3.1)), i.e. ($V = V_{\dot{m}}; \alpha = \alpha_e; \gamma = \gamma_{\dot{m}}; q = q_e$). Moreover, each moment corresponds to an aircraft total mass m_i and m_f :

$$\left\{ \begin{array}{lcl} t_i = 2.5 \text{ h} & \rightarrow & m_i = 115.53 \text{ tons} \\ t_f = 5 \text{ h} & \rightarrow & m_f = 102.67 \text{ tons} \end{array} \right.$$

The numerical simulation gives:

$$R_{\dot{m}} = 2231.08 \text{ km}$$

In addition, when the correctiong factor $\tilde{\epsilon}_{\delta x}$ is applied to the thrust lever (Eq. (19)), the numerical simulation gives in this case:

$$\widetilde{R}_e = 2237.21 \text{ km}$$

These numerical results can be compared with the result given by the Breguet range formula (Eq. (16)) for each case. In the first case, the formula is given by:

$$R_{\dot{m}}^{[B]} = \frac{V_{\dot{m}} f_e}{g c_T} \ln\left(\frac{m_i}{m_f}\right) \tag{31}$$

And for the second case:

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$$R_e^{[B]} = \frac{V_e f_e}{g c_T} \ln\left(\frac{m_i}{m_f}\right) \tag{32}$$

Based on these analytical expressions, the computation gives:

$$\begin{cases} R_{\dot{m}}^{[B]} = 2244.05 & \rightarrow \quad \dfrac{R_{\dot{m}} - R_{\dot{m}}^{[B]}}{R_{\dot{m}}^{[B]}} = -0.58\% \\ R_e^{[B]} = 2250.19 & \rightarrow \quad \dfrac{\widetilde{R}_e - R_e^{[B]}}{R_e^{[B]}} = -0.58\% \end{cases} \tag{33}$$

4.2 Analytical analysis

Considering the expression of $V_{\dot{m}}$ (Eq. (27)), we obviously have the relation:

$$\boxed{R_{\dot{m}}^{[B]} = (1 - \epsilon_V)\, R_e^{[B]}}$$

Moreover, the range is given by:

$$R = -\int_{m_i}^{m_f} \frac{V}{c_T T} \mathrm{d}m$$

Assuming a cruise at constant TAS $V_{\dot{m}}$, we have for $R_{\dot{m}}$:

$$R_{\dot{m}} = -V_{\dot{m}} \int_{m_i}^{m_f} \frac{\mathrm{d}m}{c_T T_{\dot{m}}} \tag{34}$$

With (Eq. (23)) and considering that at a given weight we have $D_{\dot{m}} = D_e = \dfrac{mg}{f_e}$, this leads to:

$$T_{\dot{m}} = \frac{mg}{f_e}\left(1 + f_e \gamma_{\dot{m}}\right)$$

The integration of (Eq. (34)) gives:

$$R_{\dot{m}} = \frac{V_{\dot{m}} f_e}{c_T g} \frac{1}{(1 + f_e \gamma_{\dot{m}})} \ln\left(\frac{m_i}{m_f}\right)$$

As $k_e = f_e \gamma_{\dot{m}}$ is small, this finally leads to:

$$\boxed{R_{\dot{m}} = (1 - k_e)\, R_{\dot{m}}^{[B]}}$$

Considering the range $\widetilde{R}_e$ performed at constant TAS V_e which takes into account the thrust lever correction $\tilde{\epsilon}_{\delta x}$, we have in this case:

$$\widetilde{R}_e = -V_e \int_{m_i}^{m_f} \frac{\mathrm{d}m}{c_T \widetilde{T}_e} \tag{35}$$

With:

$$\widetilde{T}_e = T_e\left(1 + \tilde{\epsilon}_{\delta x}\right) = T_e\left(1 + f_e \gamma_{\dot{m}}\right)$$

The integration of (Eq. (35)) gives:

$$\widetilde{R}_e = \frac{V_e f_e}{c_T g} \frac{1}{(1 + f_e \gamma_{\dot{m}})} \ln\left(\frac{m_i}{m_f}\right)$$

Which also leads to:

$$\boxed{\widetilde{R}_e = (1 - k_e)\, R_e^{[B]}}$$

Based on these analytical expressions, the computation gives:

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$$\left\{ \begin{array}{lcl} R_{\dot{m}} & = & 2230\text{km} \\ \widetilde{R}_e & = & 2236.06\text{km} \end{array} \right. \tag{36}$$

These analytical expressions are in good accordance with the numerical results presented in the previous paragraph. It must be noted that a slight difference remain (≈ 1 km) due to the approximation $\alpha_e \approx 0$ in the equilibrated equations (Eq. (15)).

5. Synthesis

It has been demonstrated that the range $R_e^{[B]}$ given by the traditional Breguet range equation has to be corrected to give the actual performance of an aircraft taking into account the decrease of its total weight.
The correcting factor to apply depends on the targeted range to compute. Typically, for a cruise at a given TAS V_e and lift over drag ratio f_e, the formula has to be corrected by k_e. Moreover, the thrust lever corresponding to the equilibrated conditions has to be increased by $\tilde{\epsilon}_{\delta x}$ to get exactly V_e for the cruise:

$$\left\{ \begin{array}{lcl} V & = & V_e \\ f & = & f_e \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} \widetilde{\delta x}_e & = & (1 + \tilde{\epsilon}_{\delta x})\, \delta x_e \\ \widetilde{R}_e & = & (1 - k_e)\, R_e^{[B]} \end{array} \right. \tag{37}$$

Additionally, another range can be computed considering that the thrust lever is not corrected which leads to an equilibrated speed in cruise $V_{\dot{m}}$ lower than the targeted TAS V_e. In this case, the corresponding range $R_{\dot{m}}$ can be estimated from $R_e^{[B]}$ by applying the correcting factor $(k_e + \epsilon_V)$:

$$\left\{ \begin{array}{lcl} \delta x & = & \delta x_e \\ f & = & f_e \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} V & = & V_{\dot{m}} \\ R_{\dot{m}} & = & (1 - (k_e + \epsilon_V))\, R_e^{[B]} \end{array} \right. \tag{38}$$

As ϵ_V and k_e are both positive terms and $\epsilon_V < k_e$ with $\lambda_T < 1$, the ranking between the different ranges computed in this article is as follows:

$$\boxed{R_e^{[B]} > R_{\dot{m}}^{[B]} > \widetilde{R}_e > R_{\dot{m}}} \tag{39}$$

Finally, the range $R_e^{[B]}$ provided by the traditional Breguet range formula is *the most optimistic* result compared to the actual performance of the aircraft.

6. Concluding remarks

In this article, it has been demonstrated that the fuel flow rate of the aircraft affects the computation of the trimmed conditions in cruise as well as the prediction of the range.
As far as the equilibrated flight is concerned and compared to its traditional computation, this study shows that at a given altitude and true airspeed, the equilibrated thrust lever has to be increased by a factor $\tilde{\epsilon}_{\delta x}$ to "pay" for the flight path angle $\gamma_{\dot{m}}$ resulting from the weight decrease of the aircraft and get the targeted TAS V_e in cruise.
An application of this result concerns the initial trimmed conditions for flight simulations which are traditionally computed at a constant weight. With this correction, the undue transients due to the application of in-flight fuel burn could be erased.
Another interesting result concerns the prediction given by the well-known Breguet range equation which should be corrected by the factor $k_e = \frac{-c_T g}{a_h V_e}$. As k_e is inversely proportional to V_e, the correction is small for the range performance of an airliner, around 0.6% in this case, which has to be compared to the 10% of deviation presented in the introduction of this article.
It must be noted that if this article is focused on the cruise performed at constant TAS and AoA, the same analysis could be performed for the cruise at constant altitude and TAS, and constant altitude and AoA.
As a conclusion, this article aims at providing engineers with a deeper understanding of aircraft equilibrium and the corrections which have to be applied to analytical expressions based on simplifying assumptions.

References

[1] B. Alayrac. *Mécanique de l'aviation: application des méthodes de la mécanique rationnelle à l'étude du vol et de la construction des avions*. Gauthiers-Villars, 1925.

DOI: 10.13009/EUCASS2019-117

[2] J.D. Anderson. *Introduction to Flight*. McGraw-Hill series in aeronautical and aerospace engineering. McGraw-Hill, 2000.

[3] Anon. U.s. standard atmosphere, 1976. *Superintendent of Documents*, 1976.

[4] Irving L Ashkenas. Range performance of turbojet airplanes. *Journal of the Aeronautical Sciences*, 2012.

[5] B. Etkin and L.D. Reid. *Dynamics of Flight: Stability and Control*. Wiley, 1995.

[6] J. J. Lee, S. Lukachko, I. Waitz, and A. Schafer. Historical and future trends in aircraft performance, cost, and emissions. *Annual Review of Energy and the Environment*, 26(1):167–200, 2001.

[7] J.D. Mattingly. *Aircraft Engine Design*. AIAA education series. American Institute of Aeronautics & Astronautics, 2002.

[8] É. Roux. *Avions civils à réaction: Plan 3 vues et données caractéristiques*. Editions Elodie Roux, 2007.

[9] M. Saarlas. *Aircraft Performance*. Wiley, 2006.

[10] Robert F. Stengel. *Flight dynamics*. Princeton University Press, 2004.

[11] E. Torenbeek. *Synthesis of Subsonic Airplane Design: An introduction to the preliminary design of subsonic general aviation and transport aircraft, with emphasis on layout, aerodynamic design, propulsion and performance*. Springer Netherlands, 1982.

[12] J.-C. Wanner, L. George, and J.-F. Vernet. *La mécanique du vol. Performances des avions et des engins*. Dunod, 1969.

[13] J. Wolny and R. Strzalka. Momentum in the dynamics of variable-mass systems: classical and relativistic case. *arXiv e-prints*, page arXiv:1810.03609, Oct 2018.