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On the use of the concept of Equivalent Single Stage Launcher

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Abstract

The paper presents a concept for enabling fast accurate possible performance of multi-stages launchers. This new concept relies on an analogy of a multistage launcher with a single stage launcher, hence enabling the use of the simple Tsiolkovski equation for one single stage. The method to get this analogy is fully explained in the paper, and it uses the so called "stage constant" of the stages as a first closing equation, the second system closing equations being necessarily solved by algebraic loop in order to trim the main parameters of a single stage launcher, i.e. structural index ($k=M_{dry}/M_{propellant}$) and the Isp.

The resolution of the system of a single stage launcher with only 4 input data from the staged launcher "initial mass, propulsive deltaV, payload mass and stage constant" allows the description of the main performances which is the relation of the propulsive deltaV versus the payload mass is hence directly coming from the Tsiolkovski equation.

Not curiously, the concept is also a very valuable demonstration of the effect of launcher staging, because the Equivalent Single Stage launcher can be determined with a extremely small structural index (k of the order of 0.004 kg of dry mass per kg of propellants for some of the multistage launchers analysed).

For showing the wide area of the accuracy of the similitude in performance gained by the novel concept, plots shows a quasi superposition of the two performance curves (the real staged launcher propulsive deltaV versus the payload mass, and the Equivalent Single Stage launcher deltaV versus the payload mass). Obviously, the plots shows almost perfectly identical curves.

In addition, the paper will present the performance curves for launchers similar to current available launchers based on this novel concept, and recommendations for launchers manufacturers will be provided for the elaboration of the only 4 input data needed for using this concept for each launcher.

Nomenclature

M_{dry}	*the stage mass without propellant also called as the structural mass M_s (kg)*
M_{pr}	*the usable propellant mass of the stage (kg)*
M_f	*final mass (of a stage or a launcher) (kg)*
M_i	*initial mass(of a stage or a launcher) (kg)*
M_{pl}	*payload mass of a launcher (kg)*
g_o	*9.80665 (m/s²)*
Isp	*Specific impulse (s)*
k	*Structural index* $k = M_{dry}/M_{pr}$ *(-)*
C_{stage}	*the "stage constant"* $= g_o.Isp.(1- M_{dry}/M_f)/(1+k)$ *that is a velocity (m/s) with M_f the stage final mass (including upper stages and M_{pl} and unusable propellant)*

Note in order to avoid basic confusions, one recall here some of the main terms used:

- The Structural index k is M_{dry}/M_{pr} *(dry to propellant mass)*
- The Mass ratio is M_f/M_i *(final to initial mass)*
- The Performance index is M_{pl}/M_i *(payload to initial mass, also called payload ratio)*
- The Structural coefficient is M_{dry}/M_i *where* $M_i = (M_{dry} + M_{pr})$ *Some experts use the term "inert mass fraction" and some also use "Stage structural ratio" as Structural index as well as Structural coefficient. This may be acceptable roughly when both are small.*

Here it is preferred to use the Structural index $k= M_{dry}/M_{pr}$ because it has an obvious definition that relates to a stage and relates the direct relation of stage's dry mass to its propellant mass, contrary to the Structural coefficient that has as main advantage of a bit lower value than the Structural index $k= M_{dry}/M_{pr}$

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1 Introduction

Having the performance curve of a launcher for sketching some particular strategies is sometime a mandatory need. Most engineers know how to use the simple Tsiolkovski equation for one single stage. But its application for a multistage launcher is by far much more difficult because a number of data is needed from the launcher manufacturer, and those data are not really available easily.
One presents a new concept for enabling fast accurate possible performance of multi-stages launchers that relies on an analogy with a single stage launcher,
The method to get this analogy is first fully explained.
Also, this new concept is shown as a very valuable demonstration of the effect of launcher staging, because the Equivalent Single Stage launcher can be determined with an extremely small structural index (k of few grams of dry mass per kg of propellant).
Also, recommendations for launchers manufacturers will be provided for the elaboration of the only 4 data needed for using this concept of Equivalent Single Stage launcher for each multi-stages launcher
In addition, the paper will present the performance curves for launchers classes similar to current available launchers.

2 THE MAIN LAUNCHER PERFORMANCE FOR A USER

For a given staged Launcher, the main performances for a potential user are the couples $(\Delta V, \boldsymbol{M}_{pl})$ that can be traced by the relationship $\Delta V = f(\boldsymbol{M}_{pl})$. To get such relation the traditional approach needs a lot of inputs data from the Launcher design with each stage and rocket engine detailed, but such design data are mainly not available for know-how reasons and for confidentiality. Hence it is difficult for any user to follow the traditional approach.

2.1 Reduction of the number of data

In such circumstances where the traditional approach is not practicable, it is interesting to define a "single stage Launcher" equivalent to the multi-stages Launcher in the sense that it approximates again the curve of the multi-stages Launcher $\Delta V = f(\boldsymbol{M}_{pl})$ because due to the simplicity of the approach, the number of data needed from the Launcher design can be reduced to a minimum.

2.2 Definition of a Single-stage launcher

In fact, from a unique performance point of the staged Launcher, that is one single couple $(\Delta V, \boldsymbol{M}_{pl})$ and the corresponding initial mass $\boldsymbol{M}_i$, we can define a single stage Launcher (not yet equivalent).
This one is characterized by a Structural index $k = M_{dry}/M_{pr}$ and an Isp .
The difficulty resides in the best assessment of both values of k and *Isp* for getting a equivalent single stage. This point is highlighted below.

2.2.1 Difficulty of the definition

With the given initial mass $\boldsymbol{M}_i$ and the given ΔV and a <u>fixed a priori</u> *Isp* , the Tsiolkovsky equation allows to produce the initial to final mass ratio M_i/M_f (using $\Delta V = g_0 Isp\ Ln\ M_i/M_f$).
And with the given payload mass $\boldsymbol{M}_{pl}$, thanks to the final mass decomposition: $M_f = \boldsymbol{M}_{pl} + M_{dry}$ this gives directly the Single stage launcher's Structural index of the dry mass wrt its propellant mass $k = M_{dry}/M_{pr}$. But the index k is function of the *Isp* that was fixed a priori initially.
By changing the fixed *Isp* we get a different value for the k index.
Thus it is clear that the difficulty is to find the *Isp* that at best represents the performance of the multi-stages Launcher in its entirety (not just locally as with partial derivatives). A new relation should be found in order to get the above property for closing the problem.

2.2.2 A proposed closing relation

The find is that the closing relation can be given by the subtle "stage constant" defined in the course of a simplified optimisation problem of multi-stages launcher (for a multi-stages launcher optimized within such simplified approach, all the C_{stage} of all stages are all equal, see the demonstration attached in annex below). The definition of C_{stage} is for each stage with M_f the final mass of the stage --*including the upper stages and the payload mass*-- and with $k = M_{dry}/M_{pr}$:

$$\boldsymbol{C_{stage}} = \boldsymbol{g_0 Isp}\ \frac{\boldsymbol{1 - M_{dry}/M_f}}{\boldsymbol{1 + M_{dry}/M_{pr}}} \quad \text{or} \quad \boldsymbol{C_{stage}} = \boldsymbol{g_0 Isp}\ \frac{\boldsymbol{(1 - k\ (M_i/M_f - 1))}}{\boldsymbol{1 + k}} \qquad \textbf{(1)}$$

It is to be highlighted that in the frame of a simplified optimisation of multi-stages launcher this C_{stage} is a unique common value. Thus it seems logic to use this unique common value of all the stages to derive the characteristics of an Equivalent Single Stage Launcher. Because the C_{stage} is linking index k and *Isp*, thus the C_{stage} gives the

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additional relationship for the Single Stage Launcher. This C_{stage} is a new input data that must come from the multi-stages Launcher (naturally considered as optimised in a simplified approach). It includes some design characteristics of the multi-stages Launcher (but not explicitly) and all the stages of the multi-stages Launcher are equal regarding this C_{stage}, this is the reason why the process works so perfectly.
Moreover, because for optimised multi-stages Launcher, all the stages are equal regarding this stage constant, it seems a logic consequence of the simplified optimisation problem to be able to consider an equivalence with less stages, any stage being undistinguishable with respect to others, and thus the multi-stages launcher can be effectively summarised with only one stage.

2.2.3 Using the closing relation

Thus just making the quantity of eq. 1 : $g_0 Isp \frac{(1-k\,(M_i/M_f-1))}{1+k}$ relative to the single stage Launcher equal to the unique value C_{stage} of the stages of the multi-stages Launcher (optimised in a simplified approach), one gets systematically all the features of the Equivalent Single Stage.
Structural index $k^{\sim}$ and $Isp^{\sim}$ (~ representing the values found in this simple iteration's loop for getting the equality of the two C_{stage}) and as a result all couples $(\Delta V, \boldsymbol{M_{pl}})$ can be obtained directly with a trivial use of the Tsiokolvsky equation of $\Delta V = g_0 Isp\ Ln\ \ M_i/M_f$ based on the Equivalent Single Stage.

And in a wide range around the payload mass $\boldsymbol{M_{pl}}$ taken as initial data we have a very low error (and error null for the point given initially $\boldsymbol{M_{pl}}$). This method allows giving the Equivalent Single Stage performance in a range much wider than with the partial derivatives $\partial \Delta V / \partial M_{pl}$. Partial derivatives can only follow the tangent to the curve at the point given initially $\boldsymbol{M_{pl}}$.

For initial conditions ΔV = 12 000 m/s and $\boldsymbol{M_{pl}}$ = 8000 kg, evaluations performed show in Figure 1 the correctness of this approach. It provides the performance of a 3 stages launcher optimised according to the annex, and corresponding to the last example exhibited in the annex.

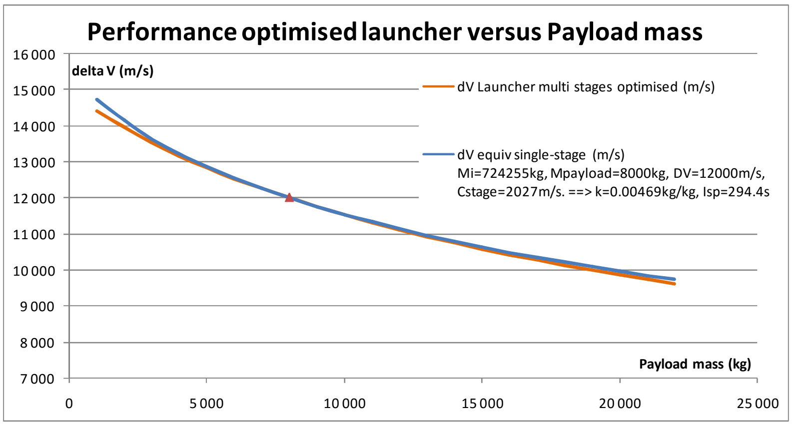

Figure 1: Heavy launcher optimised (3 stages) performance and Equivalent Single Stage concept comparisons.

It happens that the error due to this simplification is very low (blue curve comparison of single-stage Launcher equivalent and red curve for the multistage launcher optimised in a simplified approach. Thus the approximation obtained is actually "almost perfect".
Moreover, values of Structural index $k^{\sim}$ and $Isp^{\sim}$ well represent the effect of the multi-stages: the Structural index of the Equivalent Single Stage is a very very low, Structural index with $k^{\sim} \approx 0.469\%$ unlike each stages' Structural indexes of 12% or 15% and $Isp^{\sim} \approx 294.4$ s which is an intermediate value of the ones of the 3 stages.

2.2.4 Conclusions on the equivalent single-stage Launcher

When for a given Launcher, the manufacturer can give these 4 values:

$$\Delta V, \quad \boldsymbol{M_{pl}}, \quad \boldsymbol{M_i} \quad \text{and} \quad C_{stage} = g_0 Isp \frac{(1-k\,(M_i/M_f-1))}{1+k}$$

then it becomes possible to simply perform an assessment of pre-project performance of the Launcher considered for any other value of payload mass or deltaV.

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2.2.5 Recommendations for the Launcher manufacturer on the equivalent single-stage Launcher

Of course, the manufacturer has interest to give these 4 values for its multi-stages Launcher at the most of its performance (with all stages full of propellants at best).

3 CASE OF REAL STAGED LAUNCHERS

For real Launchers, the analysis of the values of the C_{stage} shows that those constant can be different from stages to stages. This is coming from considerations that are not taken into account in the simplified optimisation problem leading to the rule stating that all the C_{stage} should be equal for all the stages.
Hence several values of C_{stage} could be taken into account for the equivalent single stage.

3.1.1 Recommendations for the Launcher manufacturer on the equivalent single-stage for real Launcher

The value of C_{stage} to be taken into account in real case is an intermediate value among the one of each stages.
For finding the best C_{stage}, the Launcher manufacturer can minimise the ΔV errors on a wider range of ΔV, $\boldsymbol{M}_{pl}$, $\boldsymbol{M}_i$.

4 EXAMPLES OF EQUIVALENT SINGLE STAGE COMPARED TO REAL MULTI-STAGED LAUNCHERS PERFORMANCE

A very well known tool (PERFOL [3]) has been used to provide best estimates of the performance for several real class of launcher $\Delta V = f(\boldsymbol{M}_{pl})$. Due to some lack of genuine input data those performance are only estimates.

4.1.1 Ariane class launcher

The concept of the Equivalent Single Stage has been applied thanks to the 4 following inputs:

$$\Delta V = 11593.7\ m/s,\quad \boldsymbol{M}_{pl} = 7900\text{ kg},\quad \boldsymbol{M}_i = 767693\text{ kg}\quad\text{and}\quad C_{stage} = 1546.6\text{ m/s}$$

With the definition $C_{stage} = g_0 Isp\,\frac{(1-k\,(M_i/M_f-1))}{1+k}$, the right values for the Equivalent Single Stage of Structural index k and Isp are respectively: 0.92% and 299.8 s.
As one can see again, the effect of the staging of the real 3 stages launcher is impressive with an index for the Equivalent Single Stage of only 9.2 g of dry mass per 1 kg of propellant.
Using only the 4 input data, and the deduced values of k and Isp, and with a basic use of the Tsiolkovsky equation, one get the following performance curve Figure 2 compared to the best estimates from PERFOL [3]. For a wide range around the given payload mass, most of the errors are quite small, and the curve of the equivalent single stage performances is a little bit pessimistic with respect to the real multi-stage launcher.

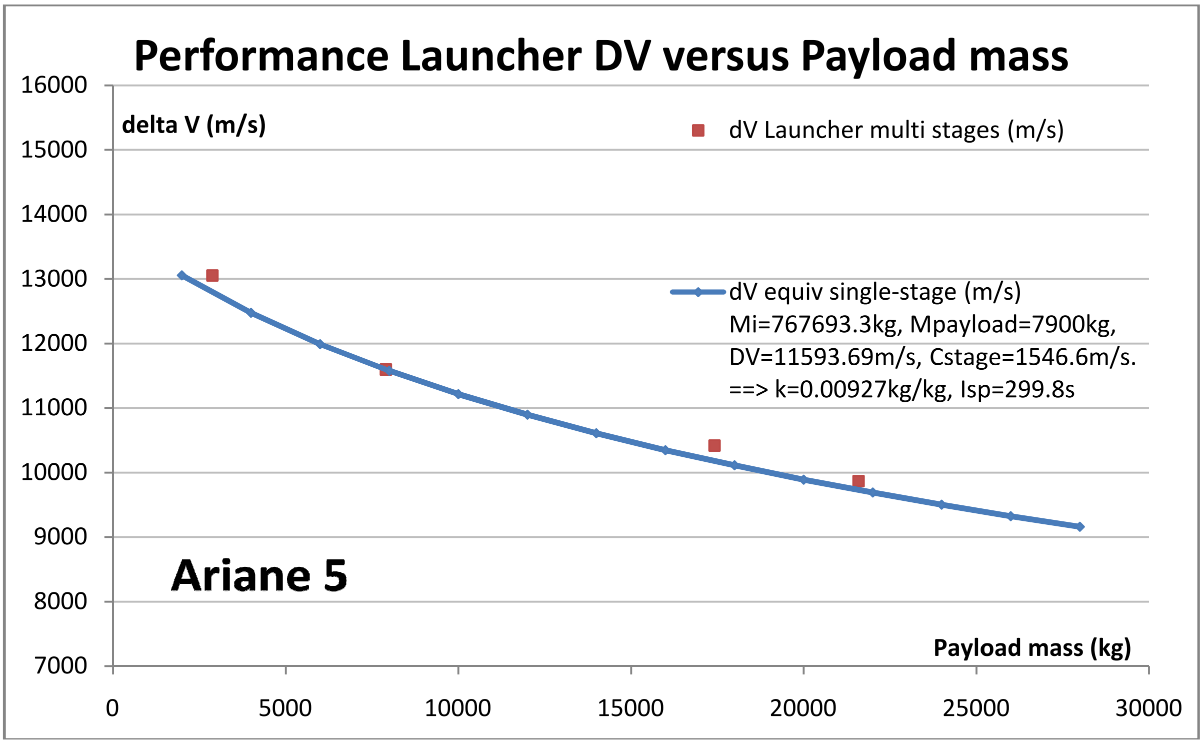

Figure 2: Ariane 5 class launcher performance obtained by Equivalent Single Stage concept, and compared to real case.

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4.1.2 VEGA class launcher

The concept of the Equivalent Single Stage has been applied thanks to the 4 following inputs:

$$\Delta V = 9440\,m/s, \quad \boldsymbol{M}_{pl} = 1345\text{ kg}, \quad \boldsymbol{M}_i = 13897\text{ kg} \quad \text{and} \quad C_{stage} = 1230.1.5\text{ m/s}$$

With the definition $C_{stage} = g_0 Isp \frac{(1-k\,(M_i/M_f-1))}{1+k}$, the right values for the Equivalent Single Stage of Structural index k and *Isp* are respectively: 0.892% and 241.1 s.

As one can see again, the effect of the staging of the real launcher is about the same as for Ariane 5 with index of only 8.9 g of dry mass per 1 kg of propellant.

Using only the 4 input data, and the deduced values of k and *Isp*, and with a basic use of the Tsiolkovsky equation, one get the following performance curve Figure 3 compared to the best estimates. For a wide range around the given payload mass, the errors are not detectable.

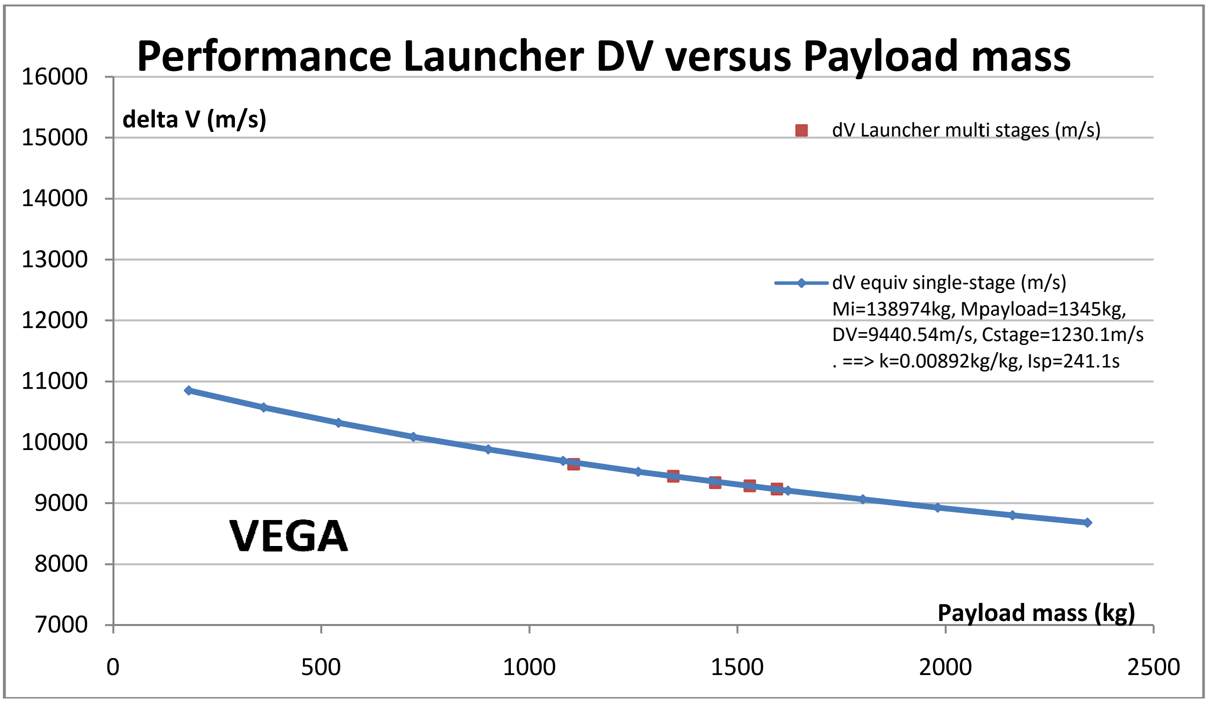

Figure 3: Vega class launcher performance obtained by Equivalent Single Stage concept, and compared to real case.

4.1.3 Soyuz class launcher

The concept of the Equivalent Single Stage has been applied thanks to the 4 following inputs:

$$\Delta V = 10310\,m/s, \quad \boldsymbol{M}_{pl} = 2115\text{ kg}, \quad \boldsymbol{M}_i = 307417\text{ kg} \quad \text{and} \quad C_{stage} = 1495.1\text{ m/s}$$

With the definition $C_{stage} = g_0 Isp \frac{(1-k\,(M_i/M_f-1))}{1+k}$, the right values for the Equivalent Single Stage of Structural index k and *Isp* are respectively: 0.35% and 230 s.

As one can see again, the effect of the staging of the real launcher is impressive with only 3.5 g of dry mass per 1 kg of propellant.

Using only the 4 input data, and the deduced values of k and *Isp*, and with a basic use of the Tsiolkovsky equation, one get the performance curve compared to the best estimates in Figure 4. For a wide range around the given payload mass, most of the errors are not detectable, except for one special point at a DV of 13.36 km/s which corresponds to the highest loading of the propellant tanks.

Normally, the concept of equivalent single-stage as presented in the paper should be applied for the highest propellant loading. However, the Soyuz case shows that even for partial loadings (due to launching constrains at Kourou) one get a really good relationship for the main performance for users.

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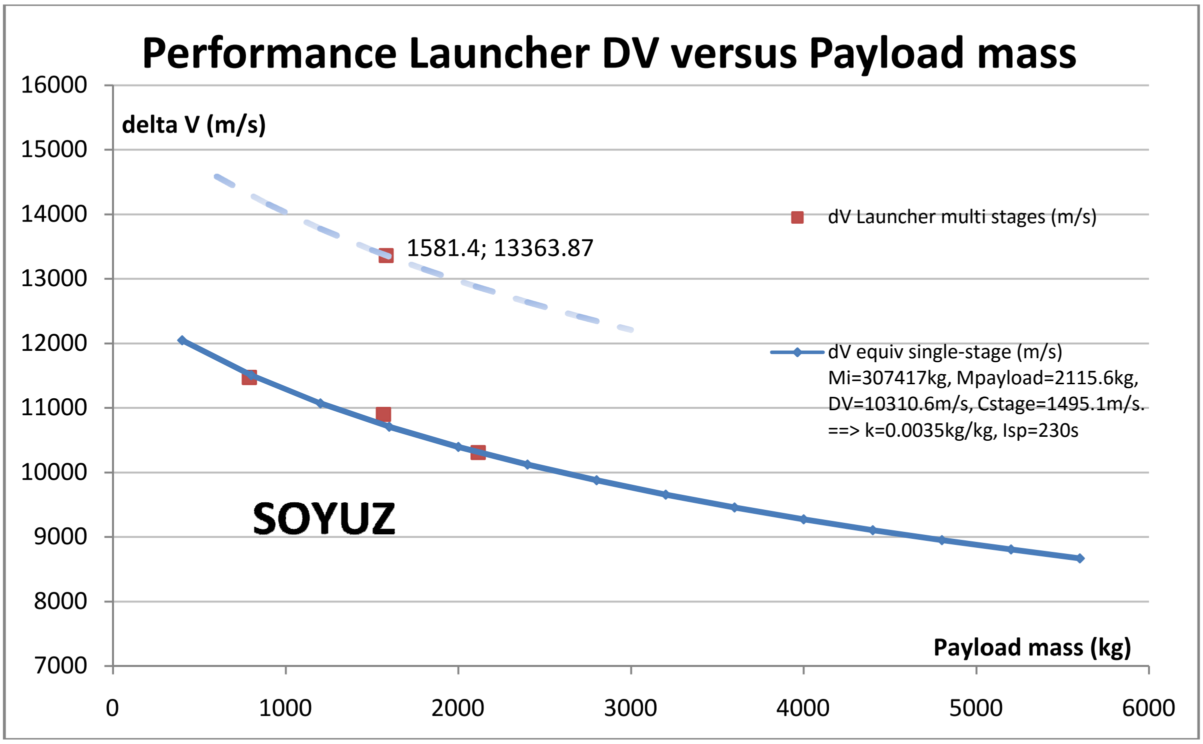

Figure 4: Soyuz class launcher performance obtained by Equivalent Single Stage concept, and compared to real case.

5 CONCLUSIONS

The main performances for users is the relationship $\Delta V = f(\boldsymbol{M_{pl}})$. Because to get such relation the traditional approach needs a lot of confidential inputs, a new concept of Equivalent Single Stage has been developed leading to a need of only 4 inputs data from the multi-stages Launcher ($\Delta V, \boldsymbol{M_{pl}}, \boldsymbol{M_i}, C_{stage}$) including the subtle "stage constant") for being able to describe with minor errors the performance of the multi-stages Launcher.

It is remarkable that only the 4 input data ($\Delta V, \boldsymbol{M_{pl}}, \boldsymbol{M_i}, C_{stage}$) enable the use of the Tsiolkovsky equation for plotting the entire performance curve of the real multi-stages Launcher.

It would be recommended to the Launcher prime manufacturer to publish the set of the 4 data as described above in order to allow potential users to work with good estimates of performance without any need of using big tools like PERFOL or other that are more reserved to flight dynamic experts.

The proposed concept is also a very good demonstration of the staging effect: the Structural index of the Equivalent Single Stage becomes a very small quantity when increasing the number of stages of the real multi-stages Launcher.

The concept of Equivalent Single Stage relies on the use of the subtle stage constant $C_{stage} = g_0 Isp \, \frac{1-M_{dry}/M_f}{1+M_{dry}/M_{pr}}$. Because for optimised multi-stages Launcher, all the stages are equal regarding this stage constant which is a unique common value, it seems a logic consequence of the simplified optimisation problem to be able to consider an Equivalent Single Stage Launcher, any stage being undistinguishable with respect to others, and thus the launcher can be effectively summarised as having only one stage.

6 ACKNOWLEDGMENT

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8 ANNEX - Exhibition of the C_{stage} from a simplified optimisation of multi-stages launcher, C_{stage} rule

Generalities[1] [2]

We consider a Launcher with *n* stages: $\boldsymbol{M_i}$ = **total** initial mass $\boldsymbol{M_{pl}}$ = payload mass.

One can get an expression for the payload mass ratio

Let : $M_{i,1} = \boldsymbol{M_i}$ and $M_{i,n+1} = \boldsymbol{M_{pl}} \Rightarrow \boldsymbol{M_{pl}}/\boldsymbol{M_i} = \frac{M_{i,n+1}}{M_{i,1}} = \frac{M_{i,n+1}}{M_{i,n}} \frac{M_{i,n}}{M_{i,n-1}} \ldots \frac{M_{i,2}}{M_{i,1}}$, so using : $\lambda_j = \frac{M_{i,j+1}}{M_{i,j}}$

one have: $\boldsymbol{M_{pl}}/\boldsymbol{M_i} = \prod_{j=1}^{n} \lambda_j$

- ΔV produced by the j^{th} stage: $\Delta V_j = g_0 Isp_j \mathrm{Ln} \frac{M_{i,j}}{M_{f,j}}$ with Isp_j the *Isp* of the j^{th} stage rocket engine,

  $M_{i,j}$ = Launcher mass when firing the j^{th} stage rocket engine;

  $M_{f,j}$= Launcher mass when turning off the rocket engine of j^{th} stage (*warning : the dry structure of the j^{th} stage $Mdry_j$ is not ejected; neither the remaining unused propellant. Those are all to be included into $Mdry_j$. Of course the mass of the upper stages (from j+1 to n+1) is to be included in such final mass of stage j*). So, using $\mu_j = \frac{M_{i,j}}{M_{f,j}}$ => $\Delta V_j = g_0 Isp_j \mathrm{Ln}\, \mu_j$

- With $k_j = \frac{Mdry_j}{Mpr_j}$ the structural index *j* where Mpr_j is the propellant mass used by the rocket engine of the j^{th} stage,

  one have $\lambda_j = \frac{1+k_j}{\mu_j} - k_j$ cf. note 1,

Optimum value process

The criteria to be optimised is $\boldsymbol{J_c} = \boldsymbol{M_{pl}}/\boldsymbol{M_i}$. The optimum staging will be given by the **max** $\boldsymbol{J_c}$ ... but as-is the problem is not enough accurate, one shall add the value of the velocity we want for the injection orbit of the payload.

The optimum staging for a given total ideal velocity increment ΔV (*one supposes that the ΔV includes all the losses, and that theses losses are known and fixed with respect to the staging: that is here that the problem is considered as simplified, and so valid only in a first approximation. But this is very characteristic of the Launcher*) will be reached for all the *n* stages together each one characterised by its μ_j, such that the criteria $\boldsymbol{J_c}$ is **maximal** , and while respecting the condition $\sum_{i=1}^{n} g_0 Isp_i \mathrm{Ln}\, \mu_i = \Delta V$

- This is an optimisation problem under constrains **(u.c.)**:. $\max_{\mu_{j:1\ldots n} \in \mathbb{R}^n - C} \; J_c(\mu_{j:1\ldots n}) \quad u.c. \quad \sum_{i=1}^{n} g_0 Isp_i Ln\, \mu_i = \Delta V$ with $\mathbb{R}^n - C$ the subset of $\mathbb{R}^n$ where the constrain ΔV is satisfied.
- One go further to a general problem without constrains by increasing the dimensions (adding a variable α) and adding to the criteria to be maximised J_c the product such that its partial derivative with respect to the variable α give zero and giving back the equation of the constrain. Hence one add to $\boldsymbol{J_c}$ the product: $\alpha(\sum_{l=1}^{n} g_0 Isp_l Ln\, \mu_l - \Delta V)$ Note: α is also named " Lagrange multiplier ".
- Hence the problem is to find $\alpha, \mu_{j:1\ldots n}$ such that one have: $\max_{\alpha, \mu_{j:1\ldots n} \in R^{n+1}} \boldsymbol{J}(\alpha, \mu_{j:1\ldots n})$

  with $\boldsymbol{J}(\alpha, \mu_{j:1\ldots n}) = \alpha(\sum_{l=1}^{n} g_0 Isp_l Ln\, \mu_l - \Delta V) + \boldsymbol{M_{pl}}/\boldsymbol{M_i}$

The optimum staging will be obtained when the last criteria is extremum, that is $\nabla \boldsymbol{J}(\boldsymbol{\alpha}, \mu_{j:1\ldots n}) = 0$ i.e. the gradient of $\boldsymbol{J}$ is the null vector i.e. one have the (n+1) following equations:

(1.) $\frac{\partial}{\partial \alpha}\left[\boldsymbol{\alpha}\left(\sum_{l=1}^{n} g_0 Isp_l Ln\, \mu_l - \Delta V\right) + \boldsymbol{M_{pl}}/\boldsymbol{M_i}\right] = 0$ *1* equation, coming from the constrain

(2.) $\frac{\partial}{\partial \mu_j}\left[\boldsymbol{\alpha}\left(\sum_{l=1}^{n} g_0 Isp_l Ln\, \mu_l - \Delta V\right) + \boldsymbol{M_{pl}}/\boldsymbol{M_i}\right] = 0$ *n* equations, for *j = 1* to *n* stages

The optimum for α and for the *n* values μ_j come from the partial derivative equations:

(1.) $= 0 \Rightarrow \sum_{l=1}^{n} g_0 Isp_l Ln\, \mu_l - \Delta V = 0$ $\sum_{l=1}^{n} g_0 Isp_l Ln\, \mu_l = \Delta V$ ==> the constrain is satisfied

(2.) $= 0 \Rightarrow \frac{\alpha g_0 Isp_j}{\mu_j} - \frac{1+k_j}{\lambda_j \mu_j^2} \boldsymbol{M_{pl}}/\boldsymbol{M_i} = 0$ for *j = 1* to *n* stages, see note 2, so $g_0 Isp_j \frac{\left(\frac{1+k_j}{\mu_j} - k_j\right)\mu_j}{1+k_j} = \frac{1}{\alpha} \boldsymbol{M_{pl}}/\boldsymbol{M_i}$

$g_0 Isp_j \frac{(1+k_j(1-\mu_j))}{1+k_j} = \frac{1}{\alpha} \boldsymbol{M_{pl}}/\boldsymbol{M_i}$ At the optimum: $\boxed{g_0 Isp_j \frac{1 - \frac{Mdry_j}{M_{f,j}}}{1 + \frac{Mdry_j}{Mpr_j}} = \frac{\mathbf{1}}{\alpha} \boldsymbol{M_{pl}}/\boldsymbol{M_i}}$ for *j = 1* to *n* stages

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1 *Note on the development of the expression of* $\lambda_j = \frac{M_{i,j+1}}{M_{i,j}}$ *... that gives* $\lambda_j = \frac{1+k_j}{\mu_j} - k_j$ *with* $\mu_j = \frac{M_{i,j}}{M_{f,j}}$ *and* $k_j = \frac{Mdry_j}{Mpr_j}$:

- *because* $M_{f,j} = M_{i,j+1} + Mdry_j$: $\frac{1}{\mu_j} + \frac{k_j}{\mu_j} = \left(\lambda_j + \frac{Mdry_j}{M_{i,j}}\right) + k_j \frac{M_{f,j}}{M_{i,j}}$ $\frac{1+k_j}{\mu_j} = \lambda_j + \frac{Mdry_j}{Mpr_j}\frac{Mpr_j}{M_{i,j}} + \frac{Mdry_j}{Mpr_j}\frac{M_{f,j}}{M_{i,j}}$
- *and because* $M_{i,j} = Mpr_j + M_{f,j}$: $\frac{1+k_j}{\mu_j} = \lambda_j + k_j\left(\frac{Mpr_j}{M_{i,j}} + \frac{M_{f,j}}{M_{i,j}}\right)$ => $\frac{1+k_j}{\mu_j} = \lambda_j + k_j$

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Optimal staging: Stage constants C_{stage} are all equal

It is defined as Stage constant (in *m/s*) the above quantity $C_{stage\ j} = g_0 Isp_j \frac{1-\frac{Mdry_j}{M_{f,j}}}{1+\frac{Mdry_j}{Mpr_j}}$. That is a characteristic speed of the stage in the launcher staging because it includes the characteristics of the stage itself but also its final mass $M_{f,j}$ (including the upper stages masses).

For an optimised launcher (in such simplified approach) **one have that they must be all equal**: $C_{stage\ j} = \frac{M_{pl}}{\alpha\ M_i}$ for *j* = *1* to *n*. because $\frac{M_{pl}}{\alpha M_i}$ is a constant that is not depending on the stages *j*.

For any stages having its Isp and index $k = M_{dry}/M_{pr}$, one can write it as well as : $C_{stage} = g_0 Isp\ \frac{1-M_{dry}/M_f}{1+M_{dry}/M_{pr}}$

or also $\boldsymbol{C_{stage} = g_0 Isp\ \frac{1-k(M_i/M_f-1)}{1+k}}$

Optimal value: solution

This kind of problem is solved by iteration: $C_{stage\ 1} = C_{stage\ 2} = \cdots \ldots .. = C_{stage\ n}$ while keeping the condition

$$\sum_{j=1}^{n} g_0 Isp_j Ln \frac{M_{i,j}}{M_{f,j}} = \Delta V$$

One can also use a solver for maximising the payload mass ratio wrt the initial mass to check the rule on the *Stage constants*.

Example :

A simple Excel sheet is used with the solver feature for maximising the payload mass ratio with respect to its launch mass ($m_{payload}/m_i$) of a generic 3 stages heavy launcher. The goal for the deltaV is 12km/s and for the payload a mass of 8 tons.

As one can see, as forecasted by the analysis above, all the stage constants are equal at the optimum (2027 m/s):

Nstages	**3**	
$m_{payload}$	**8 000**	kg
delta V goal	12 000	m/s
Launcher dV	12 000	m/s
m_i	724 255	kg
Status Optimisation	OK	
$m_i/m_{payload\ Optimised}$	**90.5**	**kg/kg**
$m_{payload}/m_i$	**1.10%**	

Stage	m_init w/o payload	m_{dry} = m_final w/o payload	$m_{propellant}$	k $m_{dry}/m_{propellant}$	isp	$m_{payload\_Stage}$	**Launcher**	m_i	m_f	dV_{Stage}	m_i/m_f	*StageConstant*	*inputs solver*
N°	kg	kg	kg		s	kg	stage N°	kg	kg	m/s	kg/kg	*m/s*	*kg*
3	56 103	7 318	48 785	**15%**	**455**	8 000	3	64 103	15 318	6 387	4.18	2 026	**48 785**
2	151 364	16 218	135 147	**12%**	**290**	64 103	2	215 467	80 320	2 806	2.68	2 027	**135 147**
1	508 788	54 513	454 275	**12%**	**290**	215 467	1	724 255	269 980	2 806	2.68	2 027	**454 275**
0	98 336	10 536	87 800	**12%**	**290**	724 255	0	822 591	734 791	321	1.12	2 503	**87 800**
-1	197 786	21 191	176 595	**12%**	**290**	822 591	-1	1 020 377	843 782	540	1.21	2 475	**-176 595**
-2	16 063	1 721	14 342	**12%**	**290**	#######	-2	1 036 440	1 022 098	40	1.01	2 535	**14 342**

[2] Because obviously $\frac{\partial}{\partial \mu_j}\left[\alpha\left(\sum_{l=1}^{n} g_0 Isp_l Ln\ \mu_l - \Delta V\right)\right] = \frac{\alpha g_0 Isp_j}{\mu_j}$ and with $\lambda_j = \frac{1+k_j}{\mu_j} - k_j$ the second term becomes:

$$\frac{\partial}{\partial \mu_j}[\boldsymbol{M}_{pl}/\boldsymbol{M}_i] = \frac{\partial}{\partial \mu_j}\prod_{l=1}^{n}\lambda_l = \left[\prod_{l=1}^{j-1}\lambda_l \prod_{l=j+1}^{n}\lambda_l\right]\frac{\partial \lambda_j}{\partial \mu_j} = \left[\frac{\prod_{l=1}^{n}\lambda_l}{\lambda_j}\right]\frac{\partial \lambda_j}{\partial \mu_j} = \frac{1}{\lambda_j}\boldsymbol{M}_{pl}/\boldsymbol{M}_i \frac{\partial}{\partial \mu_j}\left[\frac{1+k_j}{\mu_j} - k_j\right] = \frac{-(1+k_j)}{\lambda_j {\mu_j}^2}\boldsymbol{M}_{pl}/\boldsymbol{M}_i$$