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Dynamic similarity of large 3D frame structures

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Abstract
Scaled down models are widely used for experimental investigations of large structures. Usually, the real structure is so large that the available testing facilities are not adequate, or it would be too expensive. Scaling is usually carried out by simple application of the Buckingham π theorem. However, when dynamic similarity is required in structures of different materials, the scaling process is not as simple, especially if the scaled model is to be fabricated in a single material.
For complete dynamic similarity, stiffness similarity requires proper scaling of axial, bending, torsion and shear stiffness. This is not possible for general 3D frames with different materials.

1. Introduction

Scaled down models are widely used for experimental investigations of large structures. Usually, the real structure is so large that the available testing facilities are not adequate, or it would be too expensive. Scaling is usually carried out by simple application of the Buckingham π theorem. The scaled model theory was first successfully applied to the aeroelastic problem. There are some references of historical interest ([1]-[4]), classical approaches to the problem [5]-[7], a critical review [8], and later [9]-[11].
However, when dynamic similarity is required in structures of different materials, the scaling process is not as simple, especially if the scaled model is to be fabricated in a single material.
The problem to be addressed in this paper is the design of a scale model for a frame structure made up with beams of different materials. We will start with 2D pin-jointed and we will finish with 3D rigid-jointed framed structures.

2. Notation

The following symbols have been used:

- A bar section; E modulus of elasticity; l length; ρ density;
- Subscripts: m refers to the model; 0 refers to the structure.
- The subscript i indicates a generic bar/beam.
- An overline indicates a reference value of any symbol, like: $\overline{\rho A l}$, $\overline{AE/l}$, $\bar{l}$
- The λ symbol is used to specify scale factors between the model and the structure. Thus, for any x variable: $\lambda_x = (x \text{ model value} / x \text{ structure value})$

3. Truss structure

This is the simplest case study. The general equation which gives the frequencies is the one that makes:[1]

$$\det\left(-\omega^2\mathbf{M} + \mathbf{K}\right) = 0$$

The mass matrix is expressed in terms of $\rho_i A_i l_i$ of the related bars and the stiffness matrix, in terms of $A_i E_i / l_i$. Therefore we can write the aforementioned matrix as:

[1]This matrix is called Dynamic Stiffness Matrix.

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$$-\omega^2\mathbf{M}\left(\rho_i A_i l_i\right)+\mathbf{K}\left(A_i E_i / l_i\right)$$

We will take $\overline{\rho Al}$ and $\overline{AE/l}$ as reference values for the whole structure. One of them may be arbitrarily chosen. The term $\overline{\rho Al}$ is to be preferred because it is a scale factor between the model and the structure mass, and also because it has an immediate physical meaning. Dividing all terms of the mass matrix by $\overline{\rho Al}$ and those of the stiffness matrix by $\overline{AE/l}$, the system matrix obtained will be

$$-\omega^2\,\overline{\rho Al}\;\mathbf{M}\left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right)+\overline{AE/l}\;\mathbf{K}\left(\frac{A_i E_i / l_i}{\overline{AE/l}}\right)$$

The equation providing natural frequencies is:

$$\det\left(-\omega^2\,\frac{\overline{\rho Al}}{\overline{AE/l}}\mathbf{M}\left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right)+\mathbf{K}\left(\frac{A_i E_i / l_i}{\overline{AE/l}}\right)\right)=0$$

For there to be dynamic similarity, it must be verified for $\forall i$:

$$\left(\rho_i A_i l_i / \overline{\rho Al}\right)_m=\left(\rho_i A_i l_i / \overline{\rho Al}\right)_0 \qquad (1a)$$

$$\left(\frac{A_i E_i / l_i}{\overline{AE/l}}\right)_m=\left(\frac{A_i E_i / l_i}{\overline{AE/l}}\right)_0 \qquad (1b)$$

Accordingly, natural frequencies of the model and the structure are related through: $\omega_m=\omega_0\left(\lambda_{AE/l} / \lambda_{\rho Al}\right)^{1/2}$

where: $\lambda_{\rho Al}=(\overline{\rho Al})_m / (\overline{\rho Al})_0$, $\lambda_{EI/l}=(\overline{EI/l})_m / (\overline{EI/l})_0$ are the mass and stiffness scale factors, respectively.

So we can write that: $(\rho_i A_i l_i)_m=(\rho_i A_i l_i)_0\lambda_{\rho Al}$ (2a)

and $(A_i E_i / l_i)_m=(A_i E_i / l_i)_0\lambda_{AE/l}$ (2b)

The simplest case study is where all bars of the structure are of the same material and where all bars of the model are also of the same material (although the latter may be different from former).

3.1 Plane truss of homogeneous material

In this case, the features of the structure material are ρ_0, E_0 and those of the model are ρ_m, E_m. Turning to the equations (2), the sections of the model bars should be:

$$A_{mi}=A_{0i}\frac{\rho_0}{\rho_m}\lambda_{\rho Al}\lambda_l^{-1} \qquad (3)$$

in which, $\lambda_l=l_{mi} / l_{0i}$, is length scale (constant, and may be arbitrarily chosen).

The above equation allows us to calculate the areas of the model bars once the values of $\lambda_{\rho Al}$ and λ_l have been (arbitrarily) chosen. Taking this result to the second of (2), the stiffness scale factor should be:

$$\lambda_{AE/l}=\frac{\rho_0}{\rho_m}\frac{E_m}{E_0}\lambda_{\rho Al}\lambda_l^{-2} \qquad (4)$$

which is determined as a function of arbitrarily taken parameters $\lambda_l=l_{mi} / l_{0i}$. Taking this relation to the equation of frequencies similarity we obtain:

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$$\lambda_\omega = \frac{\omega_m}{\omega_0} = \left(\lambda_{AE/l} / \lambda_{\rho Al}\right)^{1/2} \qquad (5)$$

Thus, the process will be as follows:

1. Obtain the features of the structure material (E_O, ρ_O).
2. Obtain the areas of the structure bars. A_{0i}.
3. Select a material for the model $\left(E_m, \rho_m\right)$.
4. Arbitrarily choose a mass scale, $\lambda_{\rho Al}$.
5. Arbitrarily choose a length scale λ_l (usually the model will be smaller, so $\lambda_l < 1$, although it is not necessary).
6. Calculate the areas of the model bars using the equation (3): $A_{mi} = A_{0i}\left(\rho_0 / \rho_m\right)\lambda_{\rho Al}\lambda_l^{-1}$
7. Calculate the stiffness scale factor $\lambda_{AE/l}$ by means of equation (4):

$$\lambda_{AE/l} = \left(\rho_0 / \rho_m\right)\left(E_m / E_0\right)\lambda_{\rho Al}\lambda_l^{-2}$$

8. Calculate the frequency scale: $\lambda_\omega = \omega_m / \omega_0 = \left(\lambda_{AE/l} / \lambda_{\rho Al}\right)^{1/2}$

3.1.1. Example 1

Consider the structure of Figure 1. Aluminium has been chosen for the structure $\left(E_0 = 7\times10^{10}\ \text{N/m}^2, \rho_0 = 2700\ \text{kg/m}^3\right)$, and wood for the model, $E_m = 1.1\times10^{10}\ \text{N/m}^2, \rho_m = 420\ \text{kg/m}^3$. We arbitrarily take a mass scale factor $\lambda_{\rho Al} = 0.1$ and a length scale factor of $\lambda_l = 0.2$.

The bar sections are: $A_{0i} = \lfloor 1 \quad 2 \quad 3 \quad 4 \rfloor^T \times 10^{-4}\ \text{m}^2$

The area of the model bars, according to the equation (3):

$$A_{mi} = \lfloor 3.2143 \quad 6.4286 \quad 9.6429 \quad 12.857 \rfloor^T \times 10^{-4}\ \text{m}^2$$

The stiffness scale should be: $\lambda_{AE/l} = 2.5255$

Accordingly, the frequency scale should be: $\lambda_\omega = \left(\lambda_{AE/l} / \lambda_{\rho Al}\right)^{1/2} = 5.0254$

That means that natural frequencies of the model will be approximately 5 times those of the original structure.

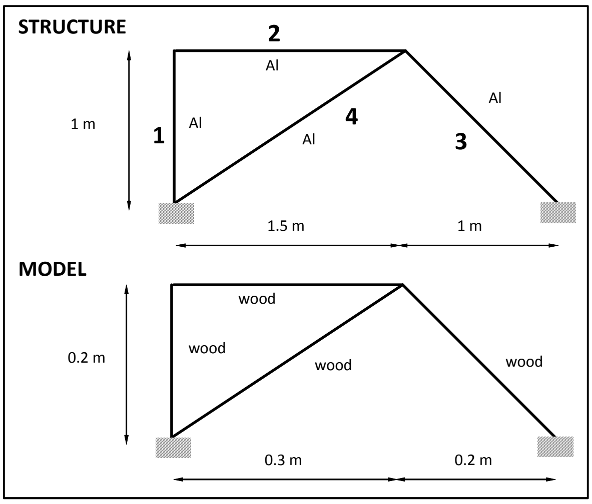

Figure 1: Example of one material, simple truss structure

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Natural frequencies of the model and those of the original structure have been solved using the MSC/Nastran programme, obtaining:

$$\text{Structure: } \lfloor 396.7082 \quad 405.8097 \quad 573.0233 \quad 794.5438 \rfloor^T \text{ Hz}$$

$$\text{Model: } \lfloor 1.9936 \quad 2.0394 \quad 2.8797 \quad 3.9929 \rfloor^T \times 10^3 \text{ Hz}$$

Whose ratio, as it can be seen, is correct.

3.2. Truss structure. Different materials

We will assume, in general, that all bars of the original structure are of different material, (E_{0i}, ρ_{0i}), and that all bars of the model are also of different materials (E_{mi}, ρ_{mi}). In this case, when we apply (3),

$$A_{mi} = A_{0i} \left(\rho_{0i} / \rho_{mi} \right) \lambda_{\rho Al} \lambda_l^{-1} .$$

However, the stiffness scale factor according to (4): $\lambda_{AE/l} = \left(\rho_{0i} / \rho_{mi} \right) \left(E_{mi} / E_{0i} \right) \lambda_{\rho Al} \lambda_l^{-2}$

Since this scale factor must be the same for the entire structure, this requires choosing materials for the model that verify the relationship: $\left(E_{mi} / \rho_{mi} \right) = \left(E_{0i} / \rho_{0i} \right)$, which is obviously not possible. Actually, it may not be a serious problem because the ratio *(E/ρ)*, which is called specific stiffness, is very similar for many materials. Here are some relationships:

Table 1: Materials behaviour

	E (GPa)	ρ $\left(\text{kg/dm}^3\right)$	E/ρ $\left(\times 10^6 \text{ m}^2\text{s}^{-2}\right)$
Aluminium	70	2.7	25.93
Steel	210	7.85	26.75
Titanium	110	4.43	24.83
Wood	11	0.42	26.19
CFRP (Carbon Fiber Reinforced Plastic)	80	1.6	50

As it can be seen, they all have similar value of *(E/ρ)*, except carbon fibre. Consequently, the stiffness scale factor can not be constant and the design of a scale model reproducing exactly the dynamic behaviour of the original structure is not possible.

To solve this issue, we proceed as follows:

1. Add to each model bar a non-structural mass uniformly distributed of value μ_{mi}, to be determined. The first equation of (3) is now written in the form: $\rho_{mi} A_{mi} + \mu_{mi} = \rho_{0i} A_{0i} \lambda_{\rho Al} \lambda_l^{-1}$
2. The value of μ_{mi} must be positive or zero, $\mu_{mi} = \rho_{0i} A_{0i} \lambda_{\rho Al} \lambda_l^{-1} - \rho_{mi} A_{mi} \geq 0$

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3. Calculate the areas of the model A_{mi} using: $A_{mi} = \left(E_{0i} / E_{mi}\right) A_{0i}\lambda_{AE/l}\lambda_l$, which can be replaced in the above equation, $\mu_{mi} = \rho_{0i}A_{0i}\lambda_{\rho Al}\lambda_l^{-1} - \rho_{mi}A_{mi} = \rho_{0i}A_{0i}\lambda_{\rho Al}\lambda_l^{-1} - \rho_{mi}\left(E_{0i} / E_{mi}\right) A_{0i}\lambda_{AE/l}\lambda_l \geq 0$ and therefore: $\rho_{0i}A_{0i}\lambda_{\rho Al}\lambda_l^{-1} - \rho_{mi}\left(E_{0i} / E_{mi}\right) A_{0i}\lambda_{AE/l}\lambda_l \geq 0$.

4. Determine the scale factor that verifies this equation: $\rho_{mi}\left(E_{0i} / E_{mi}\right) A_{0i}\lambda_{AE/l}\lambda_l \leq \rho_{0i}A_{0i}\lambda_{\rho Al}\lambda_l^{-1}$ that is to say, $\lambda_{AE/l} \leq \lambda_{\rho Al}\lambda_l^{-2}\left(E_{mi} / \rho_{mi}\right)\left(\rho_{0i} / E_{0i}\right)$, this conditions is fulfilled when

$$\lambda_{AE/l} = \lambda_{\rho Al}\lambda_l^{-2}\left(E_{mi} / \rho_{mi}\right)_{\min}\left(\rho_{0i} / E_{0i}\right)_{\min}$$

where $\left(E_{mi} / \rho_{mi}\right)_{\min}$ and $\left(\rho_{0i} / E_{0i}\right)_{\min}$ are the minimum value of quotients $\left(E_{mi} / \rho_{mi}\right)$ and $\left(\rho_{0i} / E_{0i}\right)$ of all the model and the structure materials, respectively.

5. The areas of the model bars are calculated using: $A_{mi} = \left(E_{0i} / E_{mi}\right) A_{0i}\lambda_{AE/l}\lambda_l$

6. The distributed mass is calculated by the means of:

$$\mu_{mi} = \rho_{0i}A_{0i}\lambda_{\rho Al}\lambda_l^{-1} - \rho_{mi}A_{mi} \qquad (6)$$

and it will be positive or zero for all bars.

Note:

The distributed mass μ_{mi} may be introduced into the finite element model in two ways:

· adding a non-structural mass of value μ_{mi} to each bar, given by the equation (6), or

· adding concentrated masses at the end of each bar whose value should be $\mu_{mi}l_{mi} / 2$

In the physical model, there are two ways:

· Attaching to the bars a lining of a material of low rigidity which has a mass per unit length of μ_i

· (preferable option): converting the distributed mass μ_i into two masses concentrated at the ends of each bar whose value should be: $m_i = \mu_{mi}l_{mi} / 2$, where l_{mi} is the length of each model bar. In each node, the contributions of the concurrent bars will be added.

3.3. Procedure to be followed

Ultimately, the process can be carried out following the next steps:

1. Choose, among all the structure materials, the one with the least ratio ρ_{0i} / E_{0i}.
2. Choose, among all the materials used in the model, the one with the smallest ratio E_{mi} / ρ_{mi}.
3. Determine the stiffness scale factor with the material chosen in point 1:

$$\lambda_{AE/l} = \left(\rho_0 / E_0\right)_{\min}\left(E_{mi} / \rho_{mi}\right)_{\min}\lambda_{\rho Al}\lambda_l^{-2} \qquad (7)$$

4. Determine the section of the model bar corresponding to the material chosen in point 1 by means of the equation:

$$A_{mi} = \left(\rho_0 / \rho_m\right) A_{0i}\lambda_{\rho Al}\lambda_l^{-1}$$

with ρ_0 and ρ_m the densities of the material chosen in point 1.

5. For the remaining bars, determine their section using the previously calculated stiffness scale factor:

$$A_{mi} = \left(E_{0i} / E_{mi}\right) A_{0i}\lambda_{AE/l}\lambda_l$$

6. Modify the equation (3). Calculate the distributed masses to be added to the model bars using the equation (6):

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$$\mu_{mi} = \rho_{0i} A_{0i} \lambda_{\rho Al} \lambda_l^{-1} - \rho_{mi} A_{mi}$$

This value must be positive. If this procedure is followed, it should be like this, as it will be shown in the example below.

3.3.1. Example 2

The same structure as chosen before but with different materials. We use all of them: aluminium, steel, titanium and carbon fibre. For the model, we will use wood (see figure 2).

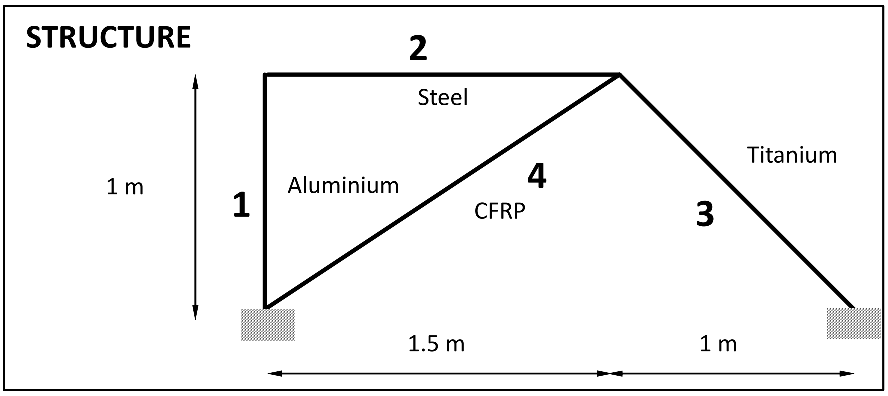

Figure 2: Example of a different materials simple truss structure

And we have to follow the steps stated above.

1. We chose as base material the carbon fibre which is the one with the lowest quotient E_O/ρ_O, which is the carbon fibre, $E_O = 8 \times 10^{10}$ MPa, $\rho_O = 1600$ kg/m³.
2. We chose wood for the whole model, $E_m = 1.1 \times 10^{10}$ MPa, $\rho_m = 420$ kg/m³.
3. We determine the stiffness scale factor using the carbon fibre as (7)

$$\lambda_{AE/l} = \left(\rho_0 / \rho_m\right)\left(E_{mi} / E_0^*\right)\lambda_{\rho Al}\lambda_l^{-2} = 1.3095$$

4. We determine the section of the bar of CFRP in the wood model: $A_m^* = 7.619 \times 10^{-4}\ \text{m}^2$
5. For the rest of the bars, determine their section using the previously calculated stiffness scale factor:

$$A_{mi} = \left(E_{0i} / E_m\right) A_{0i} \lambda_{AE/l} \lambda_l \lfloor 1.6666 \quad 9.9998 \quad 7.1427 \rfloor^T \times 10^{-4}\ \text{m}^2$$

6. Determine the mass per unit length to be added to the three bars:

$$\mu_{mi} = \rho_{0i} A_{0i} \lambda_{\rho Al} \lambda_l^{-1} - \rho_m \text{A}_{mi} = \lfloor 0.065003 \quad 0.36501 \quad 0.36451 \rfloor^T\ \text{kg/m}$$

7. We can directly introduce this non-structural mass into the MEF, or concentrate the added mass of the bars as a concentrated mass at each bar end. The mass to be located at each bar end is:

$$\mu_{mi} l_{mi} / 2 = \lfloor 0.65003 \quad 5.4752 \quad 5.1549 \rfloor^T \times 10^{-2}$$

8. The frequency scale is: $\lambda_\omega = \left(\lambda_{AE/l} / \lambda_{\rho Al}\right)^{1/2} = 3.6187$

The results obtained with MSC/Nastran are:

Structure: $\lfloor 348.55 \quad 367.55 \quad 400.33 \quad 942.44 \rfloor^T$ Hz; Model: $\lfloor 1.26 \quad 1.33 \quad 1.45 \quad 3.41 \rfloor^T \times 10^3$ Hz which, except for rounding errors, are in the correct ratio 3.7.

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4. Dynamically similar models of planar framework structure with beams

The problem of obtaining natural frequencies of a structure of beams working axially and subjected to bending stress gives rise to the following symbolic equation:

$$f\left(\omega, \rho Al, EI, AE, l\right) = 0$$

There are five parameters and three primary dimensions (*M*, *L* and *T*), so we can take two dimensionless parameters. We chose as primary parameters: ρAl, EI and l. So, we can obtain the following dimensionless factors:

$$f\left(\frac{\omega^2\left(\rho Al\right)}{\left(EI\right)}l^3, \frac{\left(AE\right)}{\left(EI\right)}l^2\left(=\frac{Al^2}{I}\right)\right)$$

$$-\omega^2\mathbf{M}\left(\rho_i A_i l_i\right) + \mathbf{K}\left(\left(AE\right)_i, \left(EI\right)_i, l_i\right)$$

Therefore, we have five parameters and we can choose three as independent to build the three basic units *(M, L, T)*. We chose the reference factors as independent: $\overline{\rho Al}, \overline{EI}$ and $\overline{l}$, and therefore the characteristic equation will be:

$$-\omega^2\frac{\overline{\rho Al}}{\overline{EI}}\overline{l}^{3}\mathbf{M}\left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right) + \mathbf{K}\left(\frac{\left(AE\right)_i}{\overline{EI}}\overline{l}^{2}, \frac{\left(EI\right)_i}{\overline{EI}}, \frac{l_i}{\overline{l}}\right)$$

For there to be dynamic similarity, all these values have to be considered, that is:

$$\left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right)_m = \left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right)_0; \quad \left(\frac{\left(AE\right)_i}{\overline{EI}}\overline{l}^{2}\right)_m = \left(\frac{\left(AE\right)_i}{\overline{EI}}\overline{l}^{2}\right)_0; \quad \left(\frac{\left(EI\right)_i}{\overline{EI}}\right)_m = \left(\frac{\left(EI\right)_i}{\overline{EI}}\right)_0; \quad \left(\frac{l_i}{\overline{l}}\right)_m = \left(\frac{l_i}{\overline{l}}\right)_0; \quad (8)$$

which, depending on the corresponding scale factors:

$$\rho_{mi}A_{mi}l_{mi} = \lambda_{\rho Al}\rho_{0i}A_{0i}l_{0i}; \quad A_{mi}E_{mi} = \lambda_{EI}\lambda_l^{-2}A_{0i}E_{0i}; \quad E_{mi}I_{mi} = \lambda_{EI}E_{0i}I_{0i}; \quad l_{mi} = \lambda_l l_{0i}$$

and the frequency scale will be: $\left(\omega^2\frac{\overline{\rho Al}}{\overline{EI}}\overline{l}^{3}\right)_m = \left(\omega^2\frac{\overline{\rho Al}}{\overline{EI}}\overline{l}^{3}\right)_0; \quad \lambda_\omega = \frac{\omega_m}{\omega_0} = \left(\lambda_{EI} / \lambda_{\rho Al} / \lambda_l^3\right)^{1/2}$

When the materials of the structure are not the same, as is clear from the previous analysis, a non-structural mass μ_{mi} should be added to each model beam. Otherwise, the stiffness scale factor λ_{EI} can not be constant for the entire structure. So, the first equation of (8) must be modified in this way

$$\left(\rho_{mi}A_{mi} + \mu_{mi}\right)l_{mi} = \lambda_{\rho Al}\rho_{0i}A_{0i}l_{0i}$$

and the following should be verified: $\left(\rho_{mi}A_{mi}l_{mi} + \mu_{mi}\right) / \left(\overline{\rho Al}\right)_m = \rho_{0i}A_{0i}l_{0i} / \left(\overline{\rho Al}\right)_0$, and it should logically be $\mu_{mi} > 0$.

In short, the ratios which must verify the properties of the model and the structure are:

$$\rho_{mi}A_{mi} + \mu_{mi} = \lambda_{\rho Al}\lambda_l^{-1}\rho_{0i}A_{0i}l_{0i}; \; E_{mi}I_{mi} = \lambda_{EI}E_{0i}I_{0i}; \; \left(A_{mi} / A_{0i}\right) = \lambda_l^{-2}\left(I_{mi} / I_{0i}\right); \; \lambda_{EI} = \left(\overline{EI}\right)_m / \left(\overline{EI}\right)_0; \quad (9)$$

is the bending stiffness scale factor. As in the previous case, the latter scale factor can not be arbitrarily taken. It has to be chosen to ensure that all values of μ_{mi} are zero or positive. To do this, the first of the (9):

$$\mu_{mi} = \lambda_{\rho Al}\lambda_l^{-1}\rho_{0i}A_{0i} - \rho_{mi}A_{mi} \quad (10)$$

From the second: $\left(I_{mi} / I_{0i}\right) = \left(E_{0i} / E_{mi}\right)\lambda_{EI}$, which has to be introduced into the third,

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$$\left(A_{mi}/A_{m0}\right)=\lambda_l^{-2}\left(I_{mi}/I_{0i}\right)=\lambda_{EI}\lambda_l^{-2}\left(E_{0i}/E_{mi}\right)$$

Finally, introducing this relationship in the first equation:

$$\mu_{mi}=\lambda_{\rho Al}\lambda_l^{-1}\rho_{0i}A_{0i}-\rho_{mi}A_{mi}=A_{0i}\rho_{0i}\lambda_l^{-1}\left(\lambda_{\rho Al}-\lambda_{EI}\lambda_l^{-1}\frac{\rho_{mi}}{\rho_{0i}}\frac{E_{0i}}{E_{mi}}\right)$$

Since the stiffness scale factor is to be chosen so that $\mu_{mi}\geq 0$, then $\lambda_{EI}\leq\lambda_{\rho Al}\lambda_i\frac{\rho_{0i}}{E_{0i}}\frac{E_{mi}}{\rho_{mi}}$.

This can be achieved by choosing: $\lambda_{EI}=\lambda_{\rho Al}\lambda_l\left(\frac{\rho_{0i}}{E_{0i}}\right)_{\min}\left(\frac{E_{mi}}{\rho_{mi}}\right)_{\min}$

As in the previous case, in this relation, $\left(\rho_{0i}/E_{0i}\right)_{\min}$ is the minimum value of the quotient *ρ/E* of all materials existing in the structure and $\left(E_{mi}/\rho_{mi}\right)_{\min}$ is the minimum value of the *E/ρ* quotient of all materials chosen for the model.

With this value λ_{EI} the rest of the parameters of the model are calculated:

$$A_{mi}=A_{m0}\frac{E_{0i}}{E_{mi}}\lambda_{EI}\lambda_l^{-2};\quad I_{mi}=I_{0i}\frac{E_{0i}}{E_{mi}}\lambda_{EI};\quad \mu_{mi}=\rho_{0i}A_{0i}\lambda_l^{-1}\lambda_{\rho Al}-\rho_{mi}A_{mi}$$

As in the previous case, μ_{mi} can be introduced as a non-structural mass added to each model beam or as two concentrated masses at the end of each bar $m_{mi}=\mu_{mi}l_{mi}/2$. This may be done in both the MEF (exactly with lumped mass model, and slightly different with coupled mass model[2]), and the physical model.

4.1. Procedure to be followed

To sum up, the procedure to be follow to build the model follows these steps:

1. Arbitrarily select the mass and length scale factors, $\lambda_{\rho Al}$ y λ_l. The mass scale factor will determine the mass of the physical model: mass of the physical model = $\lambda_{\rho Al}\times$ mass of the actual structure, and the length scale factor, the dimension: dimension of the physical model = $\lambda_l\times$ dimension of the actual structure.
2. Obtain the material properties of the structure (E_{0i},ρ_{0i}), their sections and moments of inertia of the beams, A_{0i},I_{0i} and their lengths l_{0i}.
3. Select the materials to build the model and their properties (E_{mi},ρ_{mi}). The usual practice would be choosing a single material for the whole model, but it need not necessarily be the case.
4. Determine the values of $\left(\rho_{0i}/E_{0i}\right)_{\min}$, $\left(E_{mi}/\rho_{mi}\right)_{\min}$
5. Calculate the stiffness scale factor: $\lambda_{EI}=\lambda_{\rho Al}\lambda_l\left(\rho_{0i}/E_{0i}\right)_{\min}\left(E_{mi}/\rho_{mi}\right)_{\min}$
6. Determine the sections of the model beams, $A_{mi}=\lambda_{EI}\lambda_l^{-2}A_{m0}\left(E_{0i}/E_{mi}\right)$
7. Determine the moments of inertia of the model beams $I_{mi}=\lambda_{EI}I_{0i}\left(E_{0i}/E_{mi}\right)$
8. Determine the distributed mass to be added to each model bar, $\mu_{mi}=\lambda_{\rho Al}\lambda_l^{-1}\rho_{0i}A_{0i}-\rho_{mi}A_{mi}$

   It has to be positive for all beams except for the one that produces $\left(\rho_{0i}/E_{0i}\right)_{\min}$, for which it will be zero.
9. The frequency ratio is: $\omega_m/\omega_0=\left(\lambda_{EI}/\lambda_{\rho Al}/\lambda_l^3\right)^{0.5}$

4.1.1. Example 3

[2] Both cases, selecting it in the properties palette

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Taking the example of the structure shown at the figure 3.

The features of the materials are:
Aluminium $E_{01} = 7\times10^{10}$ N/m², $\rho_{01} = 2700$ kg/m³
Steel: $E_{02} = 21\times10^{10}$ N/m², $\rho_{02} = 7850$ kg/m³
CFRP : $E_{03} = 8\times10^{10}$ N/m², $\rho_{03} = 1600$ kg/m³

The beams of the structure have square sections of 10, 20 and 30 cm each side:

$$A_{0i} = \lfloor 1 \quad 4 \quad 9 \rfloor^T \times 10^{-4}\ \text{m}^2;\ \ I_{0i} = 1/12 \times \lfloor 1^4 \quad 2^4 \quad 3^4 \rfloor^T \times 10^{-8}\ \text{m}^4$$

We chose a single material for the model: wood ($E_m = 1.1\times10^{10}$ N/m², $\rho_m = 420$ kg/m³)

We follow the steps set above.

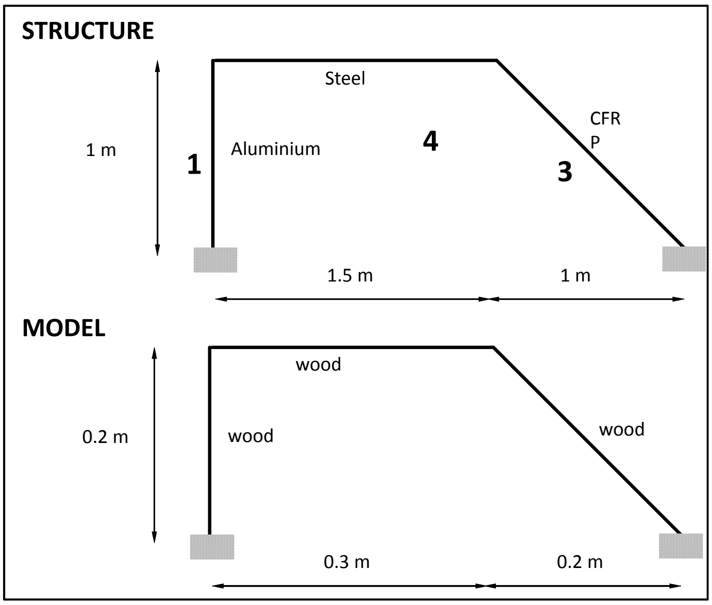

Figure 3: Example of different materials rigid jointed frame

1. Arbitrarily select the mass and length scale factors, $\lambda_{\rho Al}$ y λ_l. We chose a 1/5 scale model with 40 percent mass: $\lambda_l = 0.2$ and $\lambda_{\rho Al} = 0.4$.
Since the structure mass is $\Sigma \rho_i A_i l_i$: M = 7.0165 kg, the model mass will be M_m = 6.9165×0.4 = 2.7666 kg, as we will check later.
2. Determine the properties of the structure materials (E_{0i}, ρ_{0i}), the beams sections and moments of inertia, A_{0i}, I_{0i} and the beam lengths l_{0i}.

$$A_{0i} = \lfloor 1 \quad 4 \quad 9 \rfloor^T \times 10^{-4}\ \text{m}^2; I_{0i} = \lfloor 0.083333 \quad 1.3333 \quad 6.75 \rfloor^T \ \times 10^{-8} \text{m}^4$$

$$E_{0i} = \lfloor 7 \quad 21 \quad 8 \rfloor^T \times 10^{10}\ \text{N/m};\ \rho_{0i} = \lfloor 2700 \quad 7850 \quad 1600 \rfloor^T\ \text{kg/m}^3;\ l_{0i} = \lfloor 1 \quad 1.5 \quad 2^{1/2} \rfloor^T\ \text{m}$$

3. Choose the materials to build the model and its properties (E_{mi}, ρ_{mi}). Taking wood for the whole model $E_m = 1.1\times10^{10}\ \text{N/m}^2$, $\rho_m = 420\ \text{kg/m}^2$
4. Determine the values of $\left(\rho_{0i} / E_{0i}\right)_{\min}$, $\left(E_{mi} / \rho_{mi}\right)_{\min}$. From data, clearly it is the carbon fibre, $\left(\rho_{0i} / E_{0i}\right)_{\min} = 2\times10^{-8}$. For the model, there is no doubt: $\left(E_{mi} / \rho_{mi}\right)_{\min} = 381.\widehat{81}\times10^{-10}$
5. Calculate the stiffness scale factor $\lambda_{EI} = 4.1905\times10^{-2}$

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6. Determine the sections of the model beams, $A_{mi} = \lfloor 6.\widehat{6} \quad 80 \quad 68.572 \rfloor^T \times 10^{-4}\ \mathrm{m}^2$

7. Determine the moments of inertia of the model beams: $I_{im} = \lfloor 2.\widehat{2} \quad 106.\widehat{6} \quad 205.72 \rfloor^T \times 10^{-10}\ \mathrm{m}^4$

8. Determine the concentrated masses to be located at the end of each beam: $m_{mi} = \mu_{mi} l_{mi} / 2$, where $\mu_{mi} = \lfloor 0.26 \quad 2.92 \quad 0 \rfloor^T\ \mathrm{kg/m}$.

The masses at the ends of the beams are: $m_{mi} = \lfloor 0.026 \quad 0.438 \quad 0 \rfloor^T\ \mathrm{kg}$

9. The frequencies ratio is: $\omega_m / \omega_0 = 3.6187$. The frequencies in the model will be 3.62 times those of the original structure.

4.1.2. Checking

The model mass is: 2.8066 kg. As you will see, the quotient of the model mass and the one of the structure is: (2.8066/7.0165) = 0.4, as it should be.

The natural frequencies (obtained with Nastran) are:

Structure: $\lfloor 9.4891 \quad 266.89 \quad 520.22 \quad 1049.7 \rfloor^T$ Hz; Model: $\lfloor 34.338 \quad 965.8 \quad 1882.5 \quad 3798.7 \rfloor^T$ Hz

which, as it can be seen, are in the correct relation (3.6187).

4.1.3. Remarks

1. The structure and model natural modes are exactly the same, since both are affected by an arbitrary constant of proportionality.
2. We have assumed that in the original structure there are no concentrated masses. If any, the method can be reformulated to take it into account. It suffices to add the concentrated masses $m_{mi} = \lambda_{\rho Al} m_{0i}$ to the analogous positions of the model.

5. 3D-Structure of beams. General case

In this case, the most general dynamic stiffness matrix may be written as:

$$-\omega^2 \mathbf{M}\left(\rho_i A_i l_i\right) + \mathbf{K}\left(\left(AE\right)_i, \left(EI\right)_i, \left(GK\right)_i, \left(GA_S\right)_i, l_i\right)$$

where:

· $\left(GK\right)_i$ is the torsional stiffness of the beams, *G* is the shear modulus and *K* is the torsional stiffness constant.

· $\left(GA_s\right)_i$ is the shear stiffness of the beams, A_S is the shear effective area: $A_S = k_S A$, where k_S is the shear stiffness factor. In general that will have little significance, but it is easy to include it.

From those 7 parameters, we will take as independent: $\overline{\rho Al}, \overline{EI}$ and $\bar{l}$, with which we nondimensionalise the others:

$$-\omega^2 \frac{\overline{\rho Al}}{\overline{EI}} \bar{l}^3 \mathbf{M}\left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right) + \mathbf{K}\left(\frac{\left(AE\right)_i}{\overline{EI}} \bar{l}^2, \frac{\left(EI\right)_i}{\overline{EI}}, \frac{\left(GK\right)_i}{\overline{EI}}, \frac{\left(GA_S\right)_i}{\overline{EI}} \bar{l}^2, \frac{l_i}{\bar{l}}\right)$$

In order to have a complete dynamic similarity, the following equalities must be verified. Taking into account previous explanations, we will introduce from the beginning the non-structural mass to the model beams:

$$\left(\frac{\rho_i A_i l_i + \mu_i l_i}{\overline{\rho Al}}\right)_m = \left(\frac{\rho_i A_i l_i}{\overline{\rho Al}}\right)_0 \rightarrow \rho_{mi} A_{mi} + \mu_{mi} = \lambda_{\rho Al} \lambda_l^{-1} \rho_{0i} A_{0i}$$

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$$\left(\frac{(AE)_i}{\overline{EI}}\bar{l}^{-2}\right)_m = \left(\frac{(AE)_i}{\overline{EI}}\bar{l}^{-2}\right)_0 \rightarrow A_{mi} = \lambda_{EI}\lambda_l^{-2} A_{0i}E_{0i}/E_{mi}$$

$$\left(\frac{(EI)_i}{\overline{EI}}\right)_m = \left(\frac{(EI)_i}{\overline{EI}}\right)_0 \rightarrow I_{mi} / I_{0i} = \lambda_{EI} E_{0i}/E_{mi}$$

$$\left(\frac{(GK)_i}{\overline{EI}}\right)_m = \left(\frac{(GK)_i}{\overline{EI}}\right)_0 \rightarrow K_{mi} / K_{0i} = \lambda_{EI} G_{0i} / G_{mi}$$

$$\left(\frac{(GA_S)_i}{\overline{EI}}\bar{l}^{-2}\right)_m = \left(\frac{(GA_S)_i}{\overline{EI}}\bar{l}^{-2}\right)_0 \rightarrow A_{S,mi} / A_{S,0i} = \lambda_{EI}\lambda_l^{-2} G_{0i} / G_{mi}$$

$$\left(\frac{l_i}{\bar{l}}\right)_m = \left(\frac{l_i}{\bar{l}}\right)_0 \rightarrow ;\; l_{mi} = \lambda_l l_{oi}$$

When these conditions are fulfilled, the frequency scale is

$$\left(\omega^2 \frac{\overline{\rho A l}}{\overline{EI}}\bar{l}^{-3}\right)_m = \left(\omega^2 \frac{\overline{\rho A l}}{\overline{EI}}\bar{l}^{-3}\right)_0 \rightarrow \lambda_\omega = \frac{\omega_m}{\omega_0} = \left(\lambda_{EI} / \lambda_{\rho Al} / \lambda_l^3\right)^{1/2}$$

As before, the stiffness scale factor is chosen to ensure that the non-structural masses added to the model are all positive or zero, which has to be positive or zero:

$$\mu_{mi} = \left[\lambda_{\rho Al}\lambda_l^{-1}\rho_{0i} - \rho_{mi}\left(\lambda_{EI}\lambda_l^{-1}E_{0i} / E_{mi}\right)\right]A_{0i} \geq 0$$

From where: $\lambda_{EI} \leq \lambda_{\rho Al}\lambda_l\left(E_{mi} / \rho_{mi}\right)\left(\rho_{0i} / E_{0i}\right)$

This is achieved, for example, by making: $\lambda_{EI} = \lambda_{\rho Al}\lambda_l\left(E_{mi} / \rho_{mi}\right)_{\min}\left(\rho_{0i} / E_{0i}\right)_{\min}$

5.1. Procedure to be followed

The procedure follows these steps:

1. Arbitrarily select the mass and length scaling factors, $\lambda_{\rho Al}$ and λ_l. The mass scale factor will determine the mass of the physical model:
   mass of the physical model = $\lambda_{\rho Al}$ × mass of the real structure
   and the length scale factor, the dimension:
   dimension of the physical model = λ_l × dimension of the actual structure

2. Obtain the properties of the structure materials $\left(E_{0i}, G_{0i}, \rho_{0i}\right)$, the sections and moments of inertia of the beams, $\left(A_{0i}, I_{0i}\right)$, the torsional stiffness constant, K_{0i}, the effective shear areas $A_{S,0i}$, and their lengths l_{0i}.

3. Select the materials to build the model and their properties $\left(E_{mi}, G_{mi}, \rho_{mi}\right)$. The usual practice would be choosing a single material for the whole model, but it need not necessarily be the case.

4. Determine the values of $\left(\rho_{0i} / E_{0i}\right)_{\min}$ and $\left(E_{mi} / \rho_{mi}\right)_{\min}$.

5. Calculate the stiffness scale factor. $\lambda_{EI} = \lambda_{\rho Al}\lambda_l\left(\rho_{0i} / E_{0i}\right)_{\min}\left(E_{mi} / \rho_{mi}\right)_{\min}$

6. Determine the sections of the model beams,

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$$A_{mi} = \lambda_{EI}\lambda_l^{-2}A_{0i}\left(E_{0i} / E_{mi}\right) \qquad (11)$$

7. Determine the moments of inertia of the model beams

$$I_{mi} = \lambda_{EI}I_{0i}\left(E_{0i} / E_{mi}\right) \qquad (12)$$

8. Determine the torsional stiffness constants of the model sections, $K_{mi} = \lambda_{EI}K_{0i}\left(G_{0i} / G_{mi}\right)$

9. Determine the shear effective areas or the shear stiffness constants of the sections,

$$k_{S,mi} = \lambda_{EI}\lambda_l^{-2}k_{S,0i}\left(G_{0i} / G_{mi}\right)\left(A_{0i} / A_{mi}\right)$$

10. Determine the distributed mass to be added to each model bar, $\mu_{mi} = \lambda_{\rho Al}\lambda_l^{-1}\rho_{0i}A_{0i} - \rho_{mi}A_{mi}$ which will be positive for all the beams except for the one producing $\left(\rho_{0i} / E_{0i}\right)_{\min}$ for which it will be zero.

11. The frequency ratio is $\omega_m / \omega_0 = \left(\lambda_{EI} / \lambda_{\rho Al}\lambda_l^3\right)^{1/2}$

5.1.1. Example 4

To verify the above, we will consider the structure shown in Figure 4, consisting of four beams of variable tubular section.

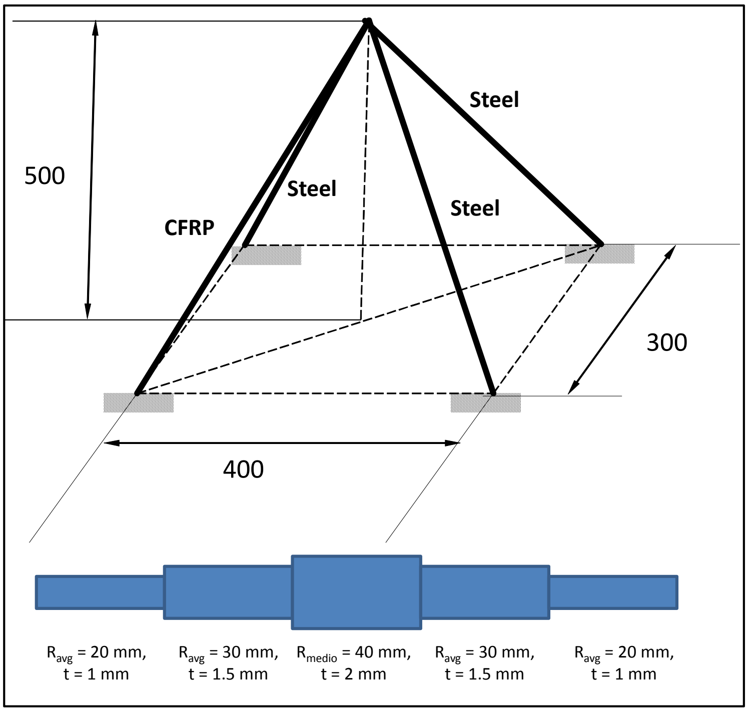

Figure 4: Example of 3D structure made of two different materials and variable cross sections

1. Arbitrarily select the mass and length scaling factors, $\lambda_{\rho Al}$ and λ_l. The mass scale factor will determine the mass of the physical model. We will choose: $\lambda_{\rho Al} = 0.4$ and $\lambda_l = 0.6$.

2. Obtain the properties of the structure materials $\left(E_{0i}, G_{0i}, \rho_{0i}\right)$. One of the beams is CFRP (E = 80 GPa, ρ = 1600×10^{-12} t/mm³, ν = 0.3) and the remaining three of steel, (E = 190 GPa, ρ = 7850×10^{-12} t/mm³, ν = 0.28).
the sections and moments of inertia of the beams, $\left(A_{0i}, I_{0i}\right)$ the torsion constants, K_{0i}, the effective shear areas $A_{S,0i}$, and their lengths l_{0i}:

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$$A_{0i} = \pi\left[\left(R_{avg} + t/2\right)^2 - \left(R_{avg} - t/2\right)^2\right];\ I_{0i} = \pi\left[\left(R_{avg} + t/2\right)^4 - \left(R_{avg} - t/2\right)^4\right]/4$$
$$G_{0i} = \pi\left[\left(R_{avg} + t/2\right)^4 - \left(R_{avg} - t/2\right)^4\right]/2;\ k_{s,0i} = 0.5;\ l_{0i} = 111.803 \text{ mm}$$

3. Select the materials to build the model and their properties $\left(E_{mi}, G_{mi}, \rho_{mi}\right)$. The usual practice would be choosing a single material for the whole model, but it need not necessarily be the case. For the model, we chose an aluminium alloy, whose properties are: (E_m = 72 GPa, ρ_m= 2700×10^{-12} t/mm^3, ν_m = 0.33)

4. Determine the values of $\left(\rho_{0i} / E_{0i}\right)_{\min}$ and $\left(E_{mi} / \rho_{mi}\right)_{\min}$
5. Calculate the stiffness scale factor $\lambda_{EI} = 0.128$
6. Determine the sections of the model beams, $A_{mi} = \lambda_{EI}\lambda_l^{-2}A_{0i}\left(E_{0i} / E_{mi}\right)$
For the structure of steel beams, $A_{mi} = 0.9382716A_{0i}$
For those of CFRP, $A_{mi} = 0.39506173A_{0i}$
7. Determine the moments of inertia of the model beams:
For steel beams, $I_{mi} = 0.128I_{0i} \times 190/72$
And for those of CFRP: $I_{mi} = 0.128I_{0i} \times 80/72$
8. Determine the torsional stiffness constant of the model sections,
For steel beams $K_{mi} = 0.35097222K_{0i}$
And for those of CFRP $K_{mi} = 0.14550427K_{0i}$
9. Determine the shear effective areas or the shear stiffness constants of the sections,
For those of steel $k_{S,mi} = 0.51953125$
And for those of CFRP, $k_{S,mi} = 0.51153846$
10. Determine the distributed mass to be added to each model bar, which will be positive for all the beams except for the one that produces $\left(\rho_{0i} / E_{0i}\right)_{\min}$ for which it will be zero.
For steel beams: $\mu_{mi} = 5.2\widehat{3}\times10^{-9}A_{0i} - 27\times10^{-10}A_{mi}$
And for those of CFRP, $\mu_{mi} = 1.0\widehat{6}\times10^{-9}A_{0i} - 27\times10^{-10}A_{mi}$, which is zero, as it should be.
11. The frequencies ratio is: $\omega_m / \omega_0 = 1.2171612$

The results obtained by MSC/Nastran for the frequencies are (Hz):

$$\text{Real} \lfloor 445.36 \quad 499 \quad 626.02 \quad 686.59 \quad 690.64 \quad 691.01 \quad 944.64 \quad 963.76 \quad 1231.50 \quad 1346.5 \rfloor^T$$
$$\text{Model} \lfloor 542.05 \quad 607.3 \quad 761.9 \quad 835.6 \quad 840.5 \quad 840.9 \quad 1149.6 \quad 1172.9 \quad 1450 \quad 1637 \rfloor^T$$

And those of the scaled model with λ_ω, (Hz):

$$\lfloor 445.34 \quad 498.97 \quad 625.95 \quad 686.48 \quad 690.54 \quad 690.90 \quad 944.51 \quad 963.62 \quad 1231.59 \quad 1346.54 \rfloor^T$$

Which gives a percentage error

$$\text{Error} = \lfloor 0.34 \quad 0.65 \quad 1.2 \quad 1.5 \quad 1.5 \quad 1.5 \quad 1.4 \quad 1.4 \quad -0.79 \quad -0.39 \rfloor^T \times 10^{-2}\,\%$$

The difference (non-zero) between the results of the actual structure and the scaled model is due to two factors:

1. The properties of the section entered into Nastran with a small number of significant figures.
2. The Poisson modules of the materials involved are not the same, which causes a (very small) difference between torsional and shear stiffness, which can not be reproduced exactly.

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In any case, it can be seen that the influence of both factors is negligible.

5.1.2. Example 5

If the previous example is modified by adding a concentrated mass of 70 kg at the apex, the changes of the model are trivial: it suffices to add a mass of 70×0.4 = 28 kg. The resulting frequencies are obtained with the corresponding scale. Indeed, the result is better than the previous case, because the beams are forced to work primarily with axial load, whose rigidity is exactly scaled.

5.1.3. All the structure beams are of the same material

In case there is only one material in the structure, the problem is quite simple, since:

1. There is no need to introduce distributed masses to adjust the stiffness scale factor.
2. The dimensions and geometry of the beams sections are subject to the same length scale factor. That is to say, if the section of the structure is tubular, for example, outer and inner radius R and r, in the model would be $\lambda_l R$ and $\lambda_l r$ respectively.
3. The only difference would be between the structure and the model Poisson modules. If they are not the same (it would be strange if they were), there will be a small scale difference in shear modulus *G*, which would affect torsional stiffness and shear stiffness. In any event, the effect would be very small.

6. Scaling factors of cross-sectional dimensions

A desirable feature of a dynamically similar model is that the cross-sections of the model are geometrically similar to those of the original structure. It is therefore appropriate to find the scale factors of the section dimensions which have to be applied in these cases.

Assume that the sections of the original structure are tubular with outer and inner radii R_{0i} and r_{0i}, respectively. The question we want to answer now is: can the cross-section be tubular as well in the model? If so, how do you get the inner and outer radii?

From the Expression (11),

$$A_{mi} = \lambda_{EI}\lambda_l^{-2} A_{0i}\left(E_{0i} / E_{mi}\right)$$

that is to say: $\pi\left(R_{mi}^2 - r_{mi}^2\right) = \lambda_{EI}\lambda_l^{-2} A_{0i}\left(E_{0i} / E_{mi}\right)$

and from the (12),

$$I_{mi} = \lambda_{EI} I_{0i}\left(E_{0i} / E_{mi}\right)$$

that is to say, $\pi\left(R_{mi}^4 - r_{mi}^4\right)/4 = \lambda_{EI} I_{0i}\left(E_{0i} / E_{mi}\right)$

From these two equations we obtain:

$$R_{mi} = \left[2\lambda_l^2\left(I_{0i} / A_{0i}\right)/\pi + \lambda_{EI}\lambda_l^{-2} A_{0i}\left(E_{0i} / E_{mi}\right)/\pi/2\right]^{1/2}$$

$$= \left[\lambda_l^2\left(R_{0i}^2 + r_{0i}^2\right)/2/\pi + \lambda_{EI}\lambda_l^{-2}\left(R_{0i}^2 - r_{0i}^2\right)\left(E_{0i} / E_{mi}\right)/2\right]^{1/2}$$

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$$r_{mi} = \left[2\lambda_l^2 \left(I_{0i} / A_{0i} \right) / \pi - \lambda_{EI} \lambda_l^{-2} A_{0i} \left(E_{0i} / E_{mi} \right) / \pi / 2 \right]^{1/2}$$

$$= \left[\lambda_l^2 \left(R_{0i}^2 + r_{0i}^2 \right) / 2 / \pi - \lambda_{EI} \lambda_l^{-2} \left(R_{0i}^2 - r_{0i}^2 \right) \left(E_{0i} / E_{mi} \right) / 2 \right]^{1/2}$$

As it can be seen, both radii scale in the model section:

1. It is not equal to the general length scale, λ_l.
2. It is different for inner and outer radii.
3. The scale of both radii depends on the material of the actual structure.

As a remarkable conclusion, note that neither the outer radius nor the inner radius of the section explicitly appears in the actual structure. This means that, whatever the geometry of the actual section, a tubular section can always be used for the model. Having said that, the section of the original structure must have identical moments of inertia, because it has been established this way in the previous explanation, where there was only a moment of inertia.

7. Conclusions

A procedure to construct dynamically similar scaled models of frame structures has been presented. In the most general 3D case, the procedure produces exact similarity of the dynamics of the structure, except for minor discrepancy arising from the impossibility of scaling properly the effect of the Poisson ratio. In any case, the difference is quite small. General, variable beam cross section as well as non-uniform material is addressed in the paper. Finally, the procedure is illustrated with a number of examples.

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