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Estimation of Regions of Attraction of Spin Modes

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Abstract

Recently developed quantitative methods for analysis of nonlinear systems are applied to spin studies. The parameters of spin modes of a general aircraft are calculated using numerical continuation on a parameter. Polynomial approximations of the reduced 5th order system with account of significant nonlinearities in aircraft aerodynamic coefficients are derived near a number of stable spin equilibrium points and validated by simulation. For the calculated polynomial approximations, low bounds of ellipsoidal approximation of regions of attraction are estimated using special optimization tools. The estimated sizes of attraction regions for different spin modes can serve as additional information about aircraft susceptibility to spin and relative danger of particular spins.

1. Introduction

Spin remains one of the most dangerous phenomena encountered in flight. Its prediction and recovery have attracted the attention of many researchers and engineers for many years. Adequate prediction of spin and investigation of susceptibility of an aircraft to spin motion are important to ensure the flight safety. Prediction of spin motion as a clear nonlinear phenomenon is based on a technique of numerical continuation of equilibrium solutions of nonlinear aircraft dynamics on a parameter and their local stability analysis [1-5]. Susceptibility to spin motion is closely related to the size of regions of attraction (ROA) of stable spin modes. Computing the exact regions of attraction for nonlinear dynamic systems is a very difficult and unsolved problem for high order systems. The significant research efforts were devoted to the estimation of the ROA invariant subsets [6-8].

Recently, significant research has been performed on the development of nonlinear analysis tools for robustness analysis of nonlinear polynomial systems, and computing regions of attraction [9–18]. These tools use polynomial sum-of-squares (SOS) optimization [18-19] and can only be applied to systems whose dynamics are described by polynomial vector fields. All previous research concerning ROA estimation was directed to analysis of normal flight regimes and validation of control system performance in a nonlinear problem definition [9-10, 17].

The idea of this paper is to apply the nonlinear ROA estimation technique to analysis of dangerous flight regimes such as spins. The size of regions of attractions in this case can be considered as a metric for estimation of the relative danger of different spin modes.

Spin equilibrium solutions depending on parameters are calculated for the 8-th order nonlinear autonomous system of ordinary differential equations, which corresponds to the six degree of freedom aircraft motion under a standard assumption of constant altitude. A traditional continuation on a parameter technique is used. A nonlinear mathematical model of aerodynamic characteristics of a generic airliner is used. The model is valid in a wide range of angles of attack and sideslip angles with account of intensive rotation effects about all three axes. The various types of wind tunnel experimental results were included in the aerodynamic model for adequate spin investigations: the results of steady aerodynamic characteristics investigations for various angles of attack and sideslip, small amplitude forced oscillations data in pitch, yaw and roll and results of rotary balance measurements.

After calculation of exact (in terms of the accepted aerodynamic model) spin equilibrium curves depending on the control surface deflections, regions of attraction of particular stable equilibrium points are estimated. For this estimation the simplified 5^{th} order equations of aircraft dynamics are used. To justify this simplification, it is shown that parameters of spin equilibria are close to each other considering the 8-th order and approximate 5^{th} order equation systems, and dynamics of perturbed motion near stable spins are also similar. For the 5^{th} order system multivariable polynomial approximations are developed around a number of calculated stable spin equilibrium points. Both, the aerodynamic model and the dynamic equations are approximated.

As computing the exact ROA is very difficult even for this simplified task formulation, a usual restriction of search to ellipsoidal approximations of the ROA is used. The lower bounds ellipsoidal approximations are computed using Lyapunov functions and recent results connecting non-negative polynomials to semi-definite programming. A special iteration procedure with 4^{th} order multivariable polynomials is used to compute Lyapunov functions and inner ellipsoidal approximations to the ROA. The analysis is performed using software available in public domain [21-23].

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The paper is arranged as follows. Aerodynamic model used for spin parameters calculation, methods and results of investigation of spin equilibrium solutions are described in Section 2. Polynomial approximation of the 5th order aircraft motion equations near stable spins are described in Section 3. Outline of regions of attraction estimation technique used in the paper, and results of the size of the ROA estimation for a set of spin modes comparing their relative danger, are given in Section 4.

2. Spin calculation

Calculation of spin modes parameters is based on a technique of numerical continuation of equilibrium solutions of nonlinear aircraft dynamics on a parameter. This requires an appropriate mathematical model of aerodynamic coefficients in a wide range of angles of attack and sideslip angles accounting possible intensive rotation. This model is essentially nonlinear.

2.1 Aerodynamic model for spin calculation

The aerodynamic model for high angles of attack conditions is usually formulated by using experimental data obtained in a wind tunnel from static, small amplitude forced oscillations (in pitch, roll and yaw) and rotary balance tests. At high angles of attack the aircraft motion with intensive rotation may strongly influence the vortical and separated flow, and the aerodynamic coefficients become strongly depend on an aircraft conical rate $\Omega = \omega b/(2V)$ where ω is angular rate, b is the wing span, and V is the magnitude of the aircraft velocity. This dependence can be described by the following way [1]:

$$C_i = C_{i_{s.r.}}(\alpha, \beta, r_a, \delta) + C_{iq_a} q_a + C_{ir_a} r_a$$

where C_i ($i = X, Y, Z, l, m, n$) are non-dimensional aerodynamic forces and moments, α, β are angles of attack and sideslip, $\delta = (\delta_e,\ \delta_a,\ \delta_r)$ are deflections of elevator, aileron and rudder, respectively, p_a, q_a, r_a are roll, pitch and yaw rate projections of angular rate on wind-body axes, respectively, connected with the body axes angular rates as follows:

$$\begin{aligned} p_a &= p\cos\alpha\cos\beta + r\sin\alpha\cos\beta + q\sin\beta \\ r_a &= p\sin\alpha - r\cos\alpha \\ q_a &= -p\cos\alpha\sin\beta - r\sin\alpha\sin\beta + q\cos\beta. \end{aligned}$$

The force and moment representation coefficient used in the work for spin parameter calculation is the following:

$$\begin{aligned} C_X &= C_X(\alpha,\beta) + C_X(\alpha, p_a) + \Delta C_X(\alpha,\delta_e) \\ C_Z &= C_Z(\alpha,\beta) + C_Z(\alpha, p_a) + \Delta C_Z(\alpha,\delta_e) \\ C_Y &= C_{Y0}(\alpha) + \Delta C_Y(\alpha,\beta) + \Delta C_Y(\alpha, p_a) + \Delta C_Y(\alpha,\delta_a) + \Delta C_Y(\alpha,\delta_r) \\ C_m &= C_{m0}(\alpha) + \Delta C_m(\alpha,\beta) + \Delta C_m(\alpha, p_a) + \Delta C_m(\alpha,\varphi) + \Delta C_m(\alpha,\delta_e) + C_{mq_a} q_a \\ C_l &= C_{l0}(\alpha) + \Delta C_l(\alpha,\beta) + \Delta C_l(\alpha, p_a) + \Delta C_l(\alpha,\delta_a) + \Delta C_l(\alpha,\delta_r) + C_{lq_a} q_a + C_{lr_a} r_a \\ C_n &= C_{n0}(\alpha) + \Delta C_n(\alpha,\beta) + \Delta C_n(\alpha, p_a) + \Delta C_n(\alpha,\delta_a) + \Delta C_n(\alpha,\delta_r) + C_{nq_a} q_a + C_{lr_a} r_a \end{aligned} \qquad (1)$$

An example of roll moment dependence on sideslip angle β and non-dimensional angular rate, showing strong nonlinear dependencies is presented in Figure 1.

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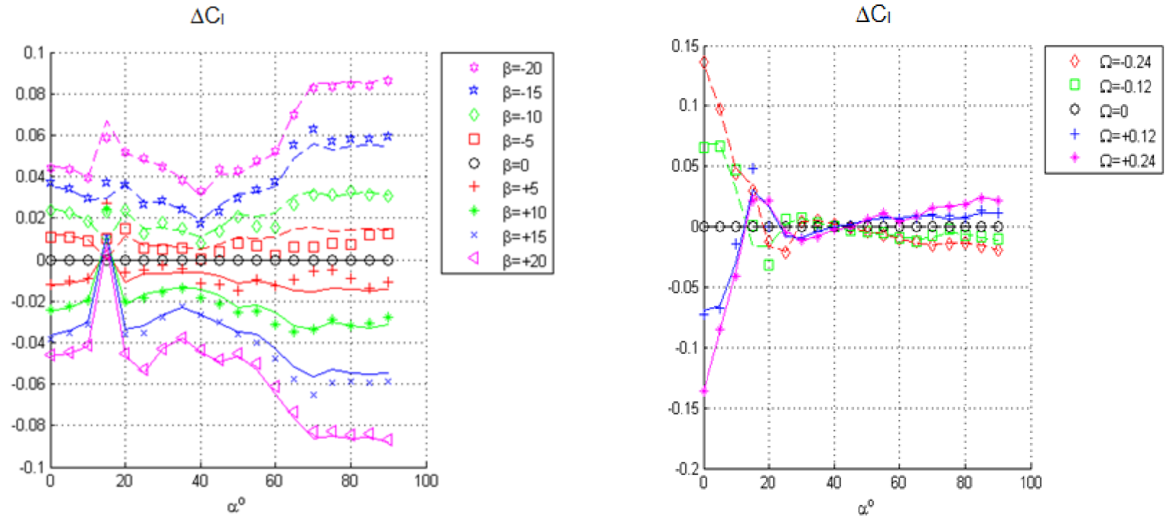

Figure 1: Roll moment dependence on sideslip angle and non-dimensional angular rate

To facilitate the implementation of the continuation technique, the nonlinear experimental dependencies $\Delta C_i(\alpha,\beta)$ ($i = Y, l, m, n$) in (1) are approximated by 3rd order polynomials of β

$$\Delta C_i(\alpha,\beta) = C_{io}(\alpha) + C_{i\beta}(\alpha)\beta + C_{i\beta^2}(\alpha)\beta^2 + C_{i\beta^3}(\alpha)\beta^3,$$

and $\Delta C_i(\alpha, p_a)$ ($i = Y, l, n$) are approximated by similar 3rd order polynomials of p_a, using standard least-square approach. In Figure 1 markers correspond to the wind tunnel experimental data and solid and dashed lines are their polynomial approximation.

2.2 Steady-state spin parameters calculation

The aerodynamic model (1) is used for computational investigation of airliner spin dynamics and further estimates. The continuation and bifurcation analysis methodologies were used effectively in flight dynamics during last four decades. A usual way to analyse the aircraft spin dynamics is considering the eighth order autonomous system of motion equations obtained from the full six-degree of freedom motion equations in an assumption of fixed altitude [1]:

$$\dot{\mathbf{x}}_8 = \mathbf{F}_8(\mathbf{x}_8, \boldsymbol{\delta}) \tag{2}$$

where the state vector is $\mathbf{x}_8 = (V, \alpha, \beta, p, q, r, \theta, \phi)' \in R^8$, and control vector includes control surface deflections $\delta = (\delta_e, \delta_a, \delta_r)' \in R^3$. The equilibrium states are defined by the system of algebraic equations:

$$\mathbf{F}_8(\mathbf{x}_8, \delta) = 0 \tag{3}$$

The equilibrium solutions of this system define an aircraft motion along a helical trajectory with vertical axis of rotation. At high angles of attack and fast rotation such solutions correspond to equilibrium spin modes. During continuation of the equilibrium solutions of the eighth order system (2) their local stability analysis using the linearized system of equations is also performed. A result of continuation of solutions of system (3) on rudder deflection δ_r as a parameter with other control deflection fixed $\delta_e = -5\deg, \delta_a = 0$, is shown in Figure 2. Stable

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equilibrium solutions are marked by red circles. Marker colors for qualitatively different types of instability are listed in Figure 2.

Analysis shows that stable spin modes exist in a wide range of control deflection parameters, in some parameter ranges there are two or more stable steady-state spins. For example, in Figure 2 there are two stable right spins in interval $25\,\text{deg} < \delta_r < 30\,\text{deg}$ and two left spins in $-22\,\text{deg} < \delta_r < -19\,\text{deg}$. Depending on the initial state values, the aircraft can enter into one of these spins. Possibility of entering into a particular spin mode and hence, a relative danger of different spins, is closely related to the size of region of attraction of this particular stable spin. However, estimation of the ROA of high order highly nonlinear systems is a tedious and non-resolved problem. Nevertheless, recent achievements in estimation of the ROA of polynomial systems [18-19, 21] allow to apply these methods to 3-5 degree polynomial systems and medium number of states. For this reason, it was decided to apply these techniques to estimation of the ROA size of stable spins using polynomial approximations of the reduced 5th order aircraft dynamics. A relation between spin parameters calculated for the 8th order and 5th order systems is presented below.

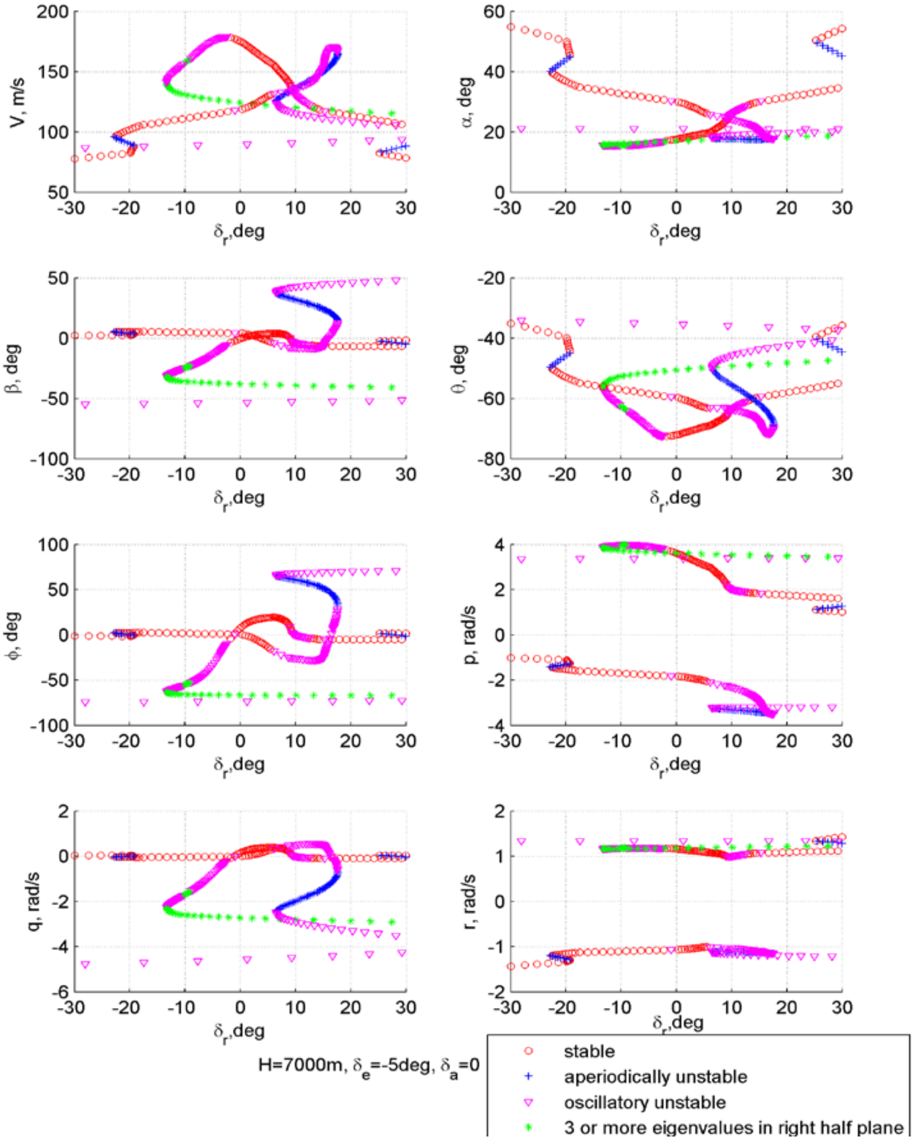

Figure 2: Equilibrium solutions for different rudder deflections, $\delta_e = -5, \delta_a = 0$

The reduced 5th order aircraft dynamics equation system is obtained from the 8th order nonlinear spatial aircraft motion neglecting the gravitational force in comparison with the inertia and aerodynamic forces (that is reasonable for fast rotations) and assuming flight speed to be constant. It has the form:

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$$\dot{\alpha} = q - (p\cos\alpha + r\sin\alpha)\tan\beta + (1/mV)(Z\cos\alpha - X\sin\alpha)/\cos\beta$$
$$\dot{\beta} = p\sin\alpha - r\cos\alpha + (1/mV)[Y\cos\beta - (X\cos\alpha + Z\sin\alpha)\sin\beta)] \quad (4)$$
$$\dot{\boldsymbol{\omega}} = J^{-1}(-\boldsymbol{\omega}\times \mathrm{J}\,\boldsymbol{\omega} + \mathbf{M}_A)$$

$$\boldsymbol{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \mathbf{M}_A = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = qS\begin{bmatrix} bC_l \\ \bar{c}C_m \\ bC_n \end{bmatrix}, X = \bar{q}SC_X, Y = \bar{q}SC_Y, Z = \bar{q}SC_Z.$$

For the reduced 5th order system (4) the same continuation procedure on a parameter δ_r is applied. Figure 3 presents one curve showing spin parameters depending on rudder deflections for the 8th order and 5th order systems together, with flight speed fixed at the value of spin equilibrium calculated for the 8th order system at $\delta_r = -30\,\text{deg}$: *V*= 77.81 m/s. It can be seen that α and β parameters coincide very well, while *p,q,r* values differ essentially. At the same time, the continuation procedure for the 5th order system was performed with variable speed *V*, changing along the 8th order spin curve. The result of this continuation is shown in Figure 3 by black line. It can be seen that this black curve coincides very closely with the original 8th order equilibrium curve. So, spin parameters calculated according to the reduced system are very close to the exact values if the velocity value of spin motion is correct.

Simulation shows that most trajectories near stable spin equilibria calculated according to the 8th order and 5th order motion equations are similar. Figure 4 shows comparison of such trajectories for both systems. Entering into spin motion from a straight-and-level flight according to the full spatial 12th order and 5th order motion equations are also reasonably similar, as shown in Figure 5.

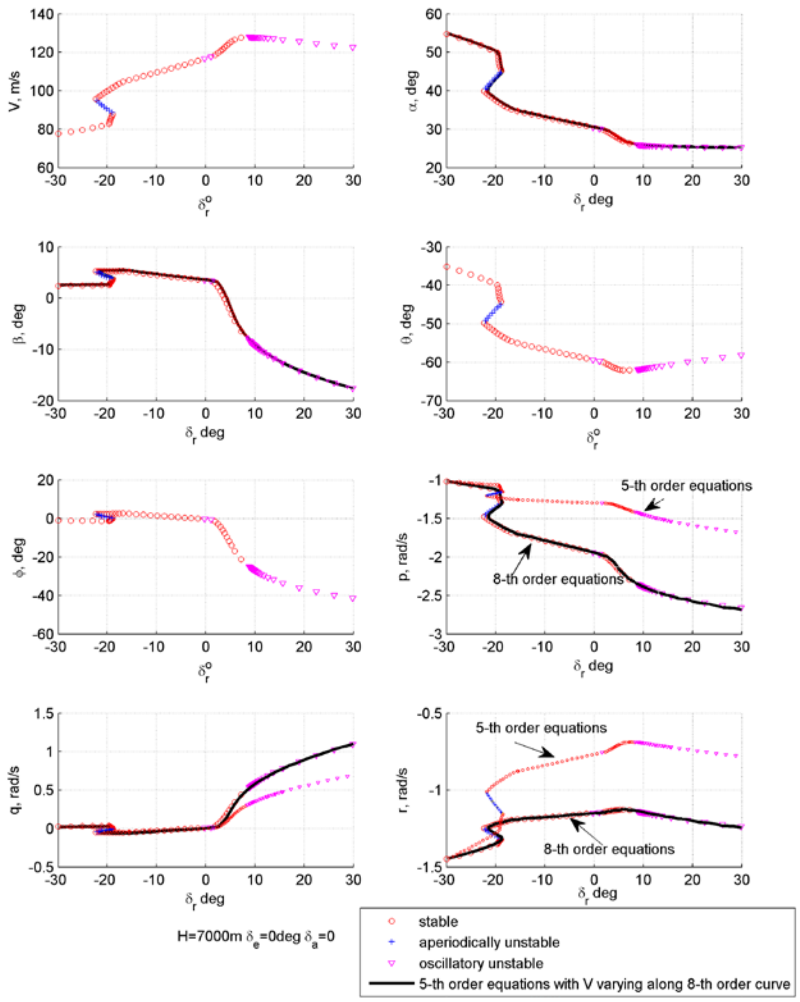

Figure 3: Equilibrium solutions for different rudder deflections.

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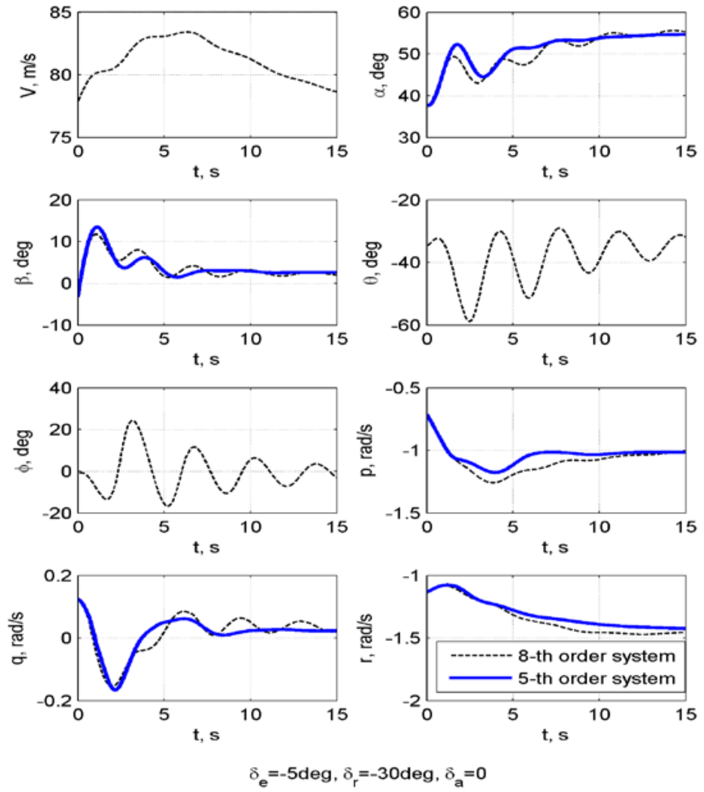

Figure 4: Comparison of trajectories near spin motion according to 8^{th} order and 5^{th} order equations

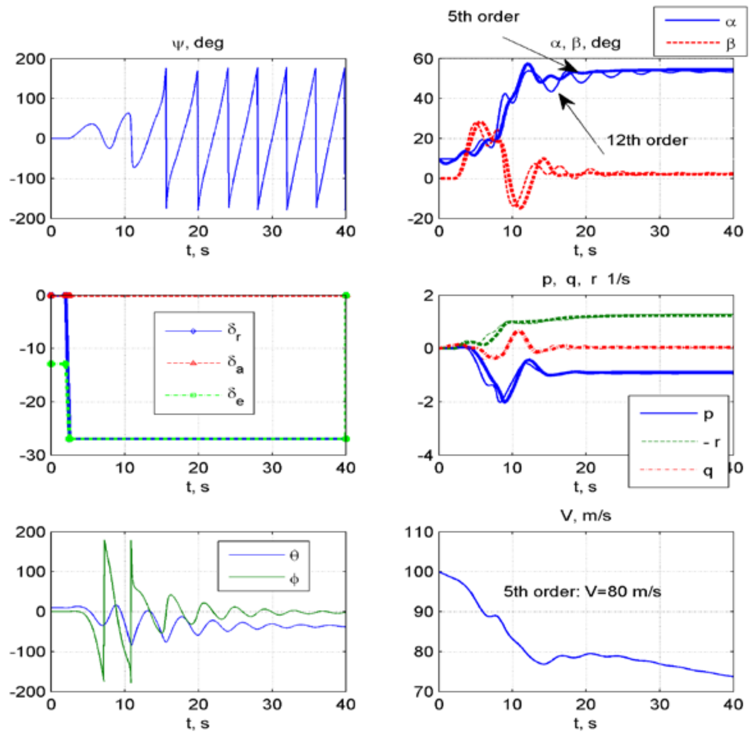

Figure5: Entering into spin motion from a straight-and-level flight: 12^{th} order and 5^{th} order equations

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3. Polynomial approximation of 5th order aircraft motion near stable spins

To construct a polynomial approximation of the 5th order aircraft motion system near a stable spin equilibrium, the nonlinear aerodynamic coefficients (1) and equations (4) were approximated by polynomials. Note, that nonlinear aerodynamic coefficients have already a 3rd degree polynomial form in sideslip and angular rate components. Equations (4) have a degree two polynomial form in angular rates. So, it is necessary to approximate the aerodynamic coefficients in alpha angle near each of the considered spin equilibrium. The least-square approximation of the look-up table data was used for fitting the data. An example of $\Delta C_i(\alpha,\beta)$ $(i = Y,l,m,n)$ and $\Delta C_i(\alpha, p_a)$ $(i = Y,l,n)$ fitting near spin equilibrium at $\delta_r = -30\deg$, $\delta_e = -5\deg$, $\delta_a = 0$, for which angle of attack is equal to $\alpha_s = 54.84\deg$ is shown in Figure 6. Sideslip angle β in this polynomial approximation ranges uniformly in interval [-20° 20°], non-dimensional angular rate $\Omega := \hat{p}_a$ ranges in interval [-0.24 0.24].

In the present paper, the following stable spin equilibrium points listed in Table 1 are considered for the ROA estimations. They are taken from two curves in Figure 2 and then adjusted according to the reduced system (4).

Table 1: List of estimated stable spins

No	δ_r, deg	V, m/s	α_s, deg	β_s, deg	p_s, rad/s	q_s, rad/s	r_s, rad/s
1	-30	77.82	54.84	2.34	-1.007	0.029	-1.433
2	-20	83.81	50.07	2.43	-1.109	0.037	-1.332
3	-20	101.80	36.76	5.34	-1.525	-0.046	-1.153
4	0	119.79	30.04	4.23	-1.828	-0.012	-1.067
5	0	174.33	17.62	-0.39	3.618	0.155	1.160
6	20	114.38	31.51	-6.59	1.752	-0.097	1.089
7	30	106.38	34.81	-6.87	1.593	-0.099	1.122
8	30	78.40	54.30	-1.78	1.023	0.047	1.425

For each equilibrium point (each α_s) a polynomial approximation around α_s is calculated separately in intervals from [-15° 15°] up to [-23° 23°] ranges, depending on the nonlinearity extent.

In addition to aerodynamic coefficient nonlinearities, system (4) is nonlinear due to trigonometric terms. Trigonometric functions of angle of attack were approximated by 3rd order polynomials for each considered stable spin mode around an appropriate angle of attack value α_s. Trigonometric functions of β were approximated by 3rd order polynomials near $\beta = 0$ since sideslip angle is small in spin motions. Fitting the equations, $C_D = -C_X \cos\alpha - C_Z \sin\alpha$ and $C_L = -C_X \sin\alpha + C_Z \cos\alpha$ data rather than C_X and C_Y were fitted for accuracy increasing. This is justified by the structure of the equations.

A degree nine polynomial model is obtained after replacing all non-polynomial terms with their polynomial approximations. The polynomial approximation to the original nonlinear model is only valid within a certain region of state-space, for different spin modes these regions differ. A result of approximation of right sides of equations (4) near spin mode No 8 (see Table 1) is shown in Figure 7. Then the polynomials were reduced to the forth order ones rejecting high order and small amplitude terms.

To validate the polynomial approximation, the numerous simulations have been performed by perturbing the states from the stable spin equilibrium conditions and compared to the original model. Most state trajectories are similar for both the polynomial and the original model. The validation approach is heuristic, however, it provides some assurance that the developed polynomial model has captured the dynamic characteristics of the original model. Figures 8 and 9 demonstrate examples of trajectories of the original system (4) and its polynomial approximations for different initial conditions. It can be seen that the 3-rd order truncated polynomial approximation gives poor coincidence, while the 4th order approximation gives rather satisfactory results.

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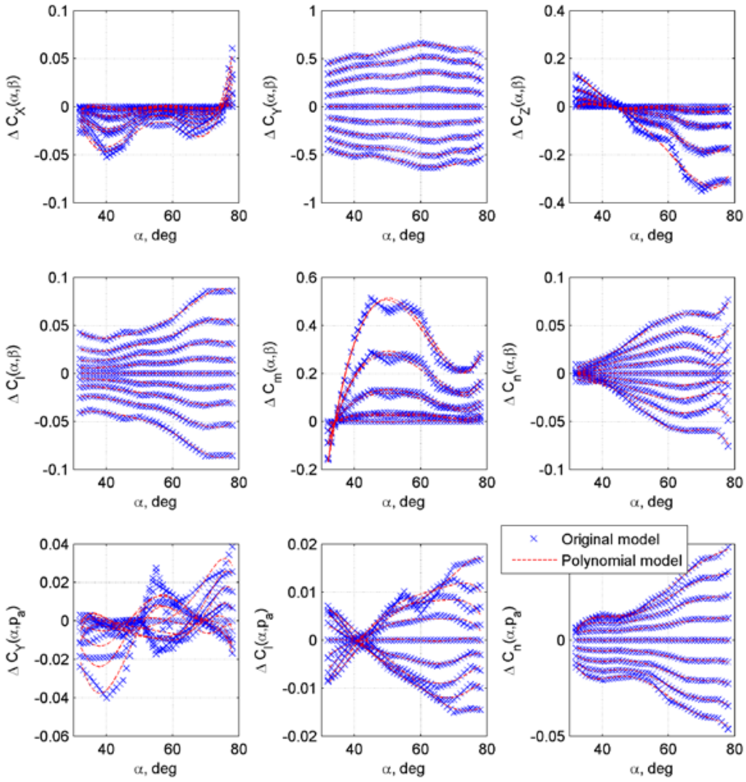

Figure 6: Look-up table data and polynomial fit for $\Delta C_i(\alpha,\beta)$ and $\Delta C_i(\alpha, p_a)$

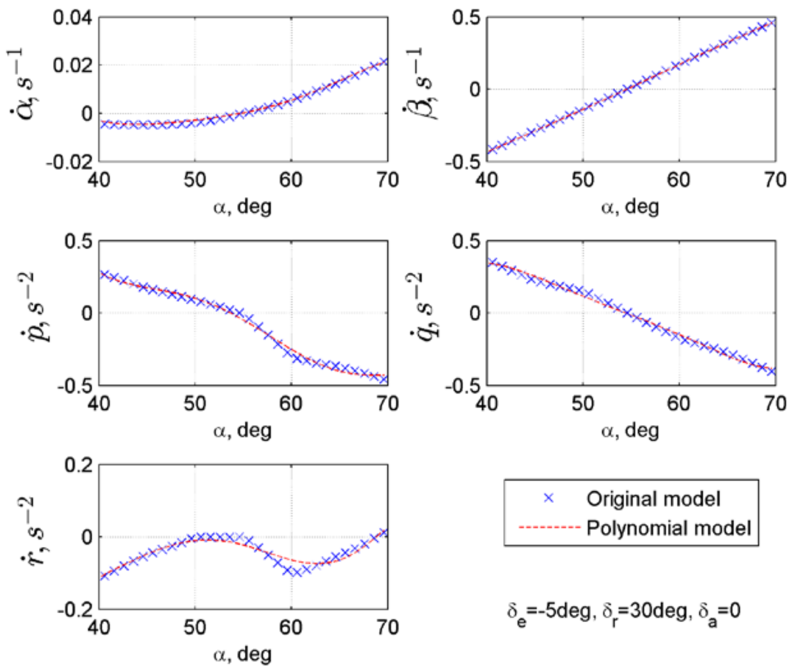

Figure 7: Original model and polynomial fit for right sides of equations (4)

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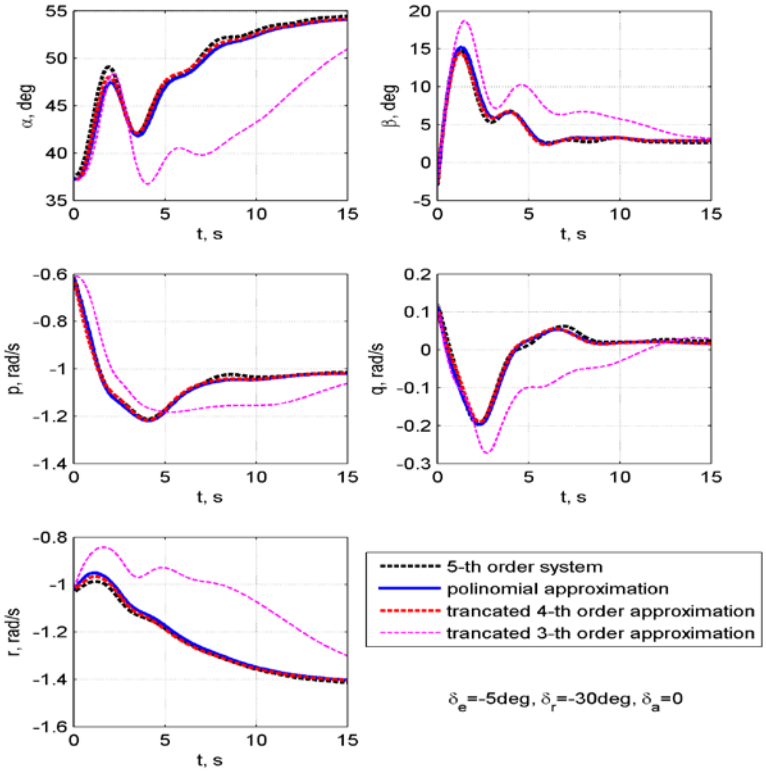

Figure 8: Comparison of time histories of motion parameters for 5th order equations and polynomial approximations

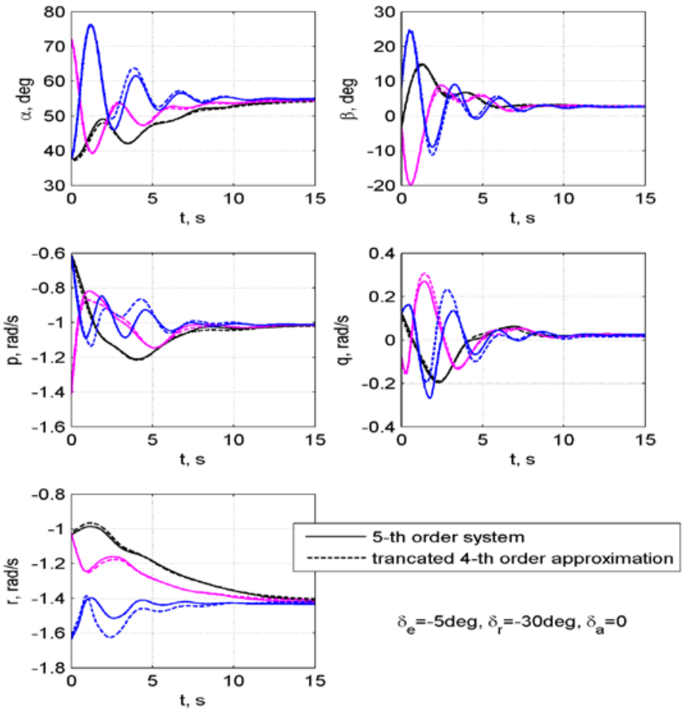

Figure 9: Comparison of time histories of motion parameters for the original 5th order equations and polynomial approximations for different initial conditions.

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4. Regions of attraction estimation

Formally, the ROA of an autonomous nonlinear dynamical system of the form

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{5}$$

where $x \in \mathrm{R}^n$ and $f : \mathrm{R}^n \to \mathrm{R}^n$ is a multivariable polynomial, is defined as follows. Assume that x_0 is an asymptotically stable equilibrium. Without loss of generality let $x_0 = 0$. The ROA is defined as

$$\Re = \left\{ x_0 \in \mathrm{R}^n : \text{if } x(0) = x_0 \text{ then} \lim_{t\to\infty} x(t) = 0 \right\}.$$

Computing the exact ROA for nonlinear dynamical systems is a very difficult task. There has been significant research devoted to estimating invariant subsets of the ROA [6-8]. The approach taken in this paper follows the approach adopted in recent years [9-17]. It supposes to restrict the search to ellipsoidal approximations of the ROA. Given an *n*x*n* matrix $N = N^T > 0$, define the shape function $p(x) := x^T N x$ and level set $\varepsilon_\beta := \left\{ x \in \mathrm{R}^n : p(x) \le \beta \right\}$ $p(x)$ defines the shape of the ellipsoid and β determines the size of the ellipsoid ε_β. The choice of p reflects the importance of certain directions in the state space. *N* is usually taken to be diagonal. Given the shape function *p*, the problem is formulated as finding the largest ellipsoid ε_β contained in the ROA:

$$\beta^* = \max \beta \quad \text{subject to} \quad \varepsilon_\beta \subset \Re .$$

Since determining the best ellipsoidal approximation to the ROA is still a sophisticated computational problem, lower and upper bounds for β^* $(\underline{\beta} \le \beta^* \le \overline{\beta})$ are computed instead. If the lower and upper bounds are close then the largest ellipsoid level set is approximately computed. The lower bounds give a guaranteeing estimate of the ROA. They are computed using Lyapunov functions and recent results connecting sums-of-squares polynomials to semi-definite programming. Computing these bounds is based on the following important theorem [20]:
If there exists $\gamma > 0$ and a polynomial Lyapunov function $V : \mathrm{R}^n \to \mathrm{R}$ such that

$$\begin{aligned} & V(0) = 0 \quad \text{and } V(x) > 0 \ \ \forall x \ne 0 \\ & \Omega_\gamma := \left\{ x \in \mathrm{R}^n : V(x) \le \gamma \right\} \quad \text{is bounded} \\ & \Omega_\gamma \subset \left\{ x \in \mathrm{R}^n : \nabla V(x) f(x) < 0 \right\} \cup \{0\} \end{aligned} \tag{6}$$

then for all $x \in \Omega_\gamma$, the solution of Equation (5) exists, satisfies $x(t) \in \Omega_\gamma$ for all $t \ge 0$, and $\Omega_\gamma \subset \Re$.
A Lyapunov function *V* satisfying these conditions, provides an estimate of the region of attraction. A first Lyapunov function candidate is a quadratic function $V_{lin} = x^T P x$ where $P > 0$ is a solution of the Lyapunov equation

$$A^T P + PA = -I, \quad A := \left. \frac{\partial f}{\partial x} \right|_{x=0}$$

This function satisfies the theorem conditions for sufficiently small $\gamma > 0$ and can be used to compute a lower bound on β^* by solving the optimizations:

$$\gamma^* := \max \gamma \quad \text{subject to} \quad \Omega_\gamma \subset \left\{ x \in \mathrm{R}^n : \nabla V(x) f(x) < 0 \right\} \tag{7}$$

$$\underline{\beta} := \max \beta \quad \text{subject to} \quad \varepsilon_\beta \subset \Omega_{\gamma^*} \tag{8}$$

For constructive solving of such optimization problems with set containment constraints it was proposed to replace them by sufficient conditions involving non-negative polynomials [13-14, 17-19]. A multivariate polynomial h(x) is non-negative if there exist such polynomials $\{g_1, \ldots g_n\}$ that $h = g_1^2 + \ldots g_n^2$ (sufficient condition). Polynomials that can be constructed in this way are called sum-of-squares (SOS) polynomials and optimization methods exploiting these

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polynomials are known as SOS-optimization. Thus, the optimization task (8) is replaced firstly by a sufficient condition

$$\underline{\beta} := \max_{\beta, s(x)} \beta \quad \text{subject to} \quad s(x) \geq 0$$
$$-(\beta - p(x))s(x) - (\gamma^* - V(x)) \geq 0 \quad \forall x$$

and then by an SOS optimization problem:

$$\underline{\beta} := \max \beta \quad \text{subject to} \quad s(x) \in \text{SOS}$$
$$-(\beta - p(x))s(x) - (\gamma^* - V(x)) \in \text{SOS}$$

where s(x) is a decision variable.
The algorithm for the ROA estimation is taken from [21, 23]. It consists in performing the so called V-s iterations:

1. γ step: hold V fixed and solve for s_2 and γ^*

   $$\gamma^* := \max_{s_2 \in SOS, \gamma} \gamma \qquad \text{subject to} \qquad -(\gamma - V)s_2 - \left(\frac{\delta V}{\delta x} f + l_2\right) \in SOS$$

2. β step: hold V, γ^* fixed and solve for s_1 and $\underline{\beta}$

   $$\underline{\beta} = \max_{s_1 \in SOS, \beta} \gamma \quad \text{subject to} \quad -(\beta - p)s_1 - \left(\gamma^* - V\right) \in SOS$$

3. V step: hold s_1, s_2, $\underline{\beta}$, γ^* fixed and solve for V satisfying:

   $$-(\gamma^* - V)s_2 - \left(\frac{\delta V}{\delta x} f + l_2\right) \in \text{SOS}$$
   $$-(\beta - p)s_1 + (\gamma^* - V) \in \text{SOS} \qquad V - l_1 \in \text{SOS} \quad V(0) = 0.$$

Here $l_1 = -\kappa_1 x^T x$, $l_2 = -\kappa_2 x^T x$, $\kappa_1 = \kappa_2 = 10^{-7} \div 10^{-6}$. Steps 1-3 are repeated as long as the lower bound $\underline{\beta}$ continues to increase. Software for V-s iterations were taken from open-source tools [21-23].
Figure 10 shows a result of 20 V-s iterations with quartic Lyapunov functions for the first spin equilibrium point from Table 1, projections on different state coordinate planes are shown. Black lines show increasing sublevels of Lyapunov function $V(x) = \gamma$, and magenta lines show increasing with iterations ellipsoids $p(x) = x^T N x$ which interior is a guaranteed part of the ROA. The center of the ellipsoid is at the spin equilibrium. Weighting matrix N in this study was taken to be identity. The length of a semiaxis of the ellipsoid along α and β direction is about 30 deg (recall $\alpha_s \simeq 55\,\text{deg}$), that is rather substantial.

Another estimation of the ROA for the same spin equilibrium was performed with matrix N=diag(5.25,5.25,1.31,1.31,1.31) supposing that region of attraction is 2.5 times larger in *p, q, r* directions than in α, β direction. The result of this estimation is shown in Figure 11 together with the previous result. In this case, the ellipsoid is much larger in the *p, q, r* directions. Note, that both ellipsoids give the guaranteed estimation of the ROA. This means that the union of these ellipsoids is also the ROA. This remark gives a way of increasing the obtained ROA estimations. It is worth to say that in multidimensional systems, regions of attraction have usually very intricate form. This can be seen from examples of cross-sections of the ROA of spin equilibrium points calculated by direct simulation in [4].
Results of similar estimations of the ROA for other spin equilibrium points listed in Table 1 are given in Figure 12. It can be seen that due to lateral aerodynamic asymmetry, right and left spins have different sizes of the ROA, and hence, different probability of entering into these spins. Comparison of the ROA for two spins existing for the same parameter values, $\delta_r = -20^o$ (regimes No 2: $\alpha_s = 50^o$, and No 3: $\alpha_s = 37^o$ in Table 1) shows that the second spin mode has the larger ROA size and hence, is more probable. Comparison of the ROA for two different spins at $\delta_r = 30^o$ (regimes No 7, $\alpha_s = 35^o$ and No 8, $\alpha_s = 54^o$) shows that in this case the ROA sizes are almost equal, and they are equally probable.
Note, that even for neutral rudder and aileron defections, and the elevator position near straight-and-level flight, there are two stable spin modes (regimes No 4: $\alpha_s = 30^o$, and No 5: $\alpha_s = 18^o$) with the notable ROA sizes, the ROA for the spin equilibrium with positive roll rate is larger than for spin with negative roll rate, that is the first spin is more probable.

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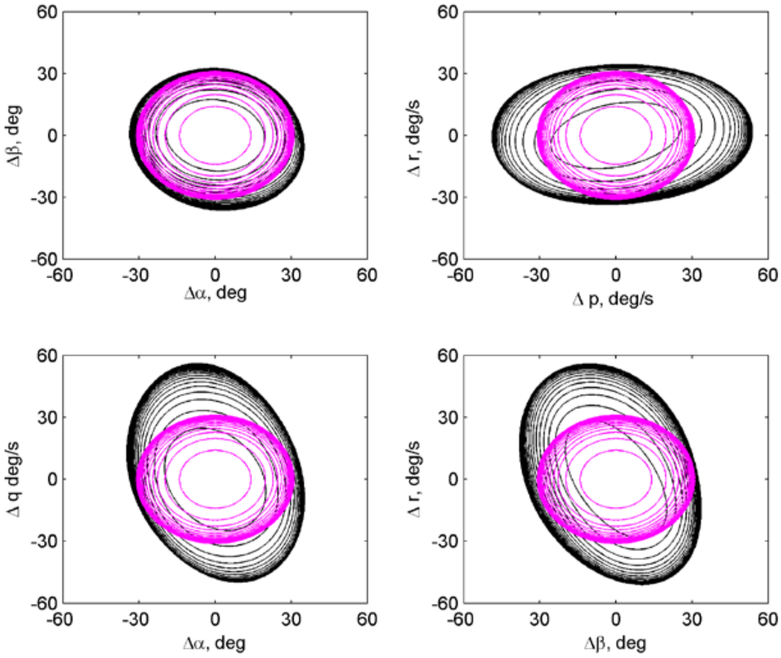

Figure 10: Projections of sublevel sets of Lyapunov functions and estimated ellipsoidal approximations

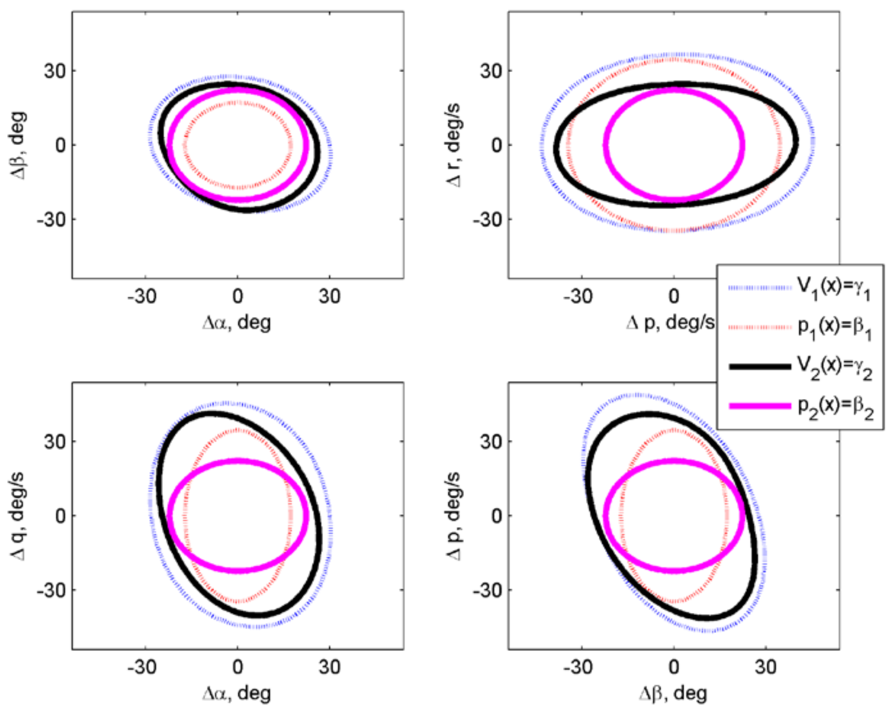

Figure 11: Projections of sublevel sets of Lyapunov functions and estimated ellipsoidal approximations for N=diag(5.25,5.25,1.31,1.31,1.31) (V_1,p_1), and N=diag(1,1,1,1,1) (V_2,p_2).

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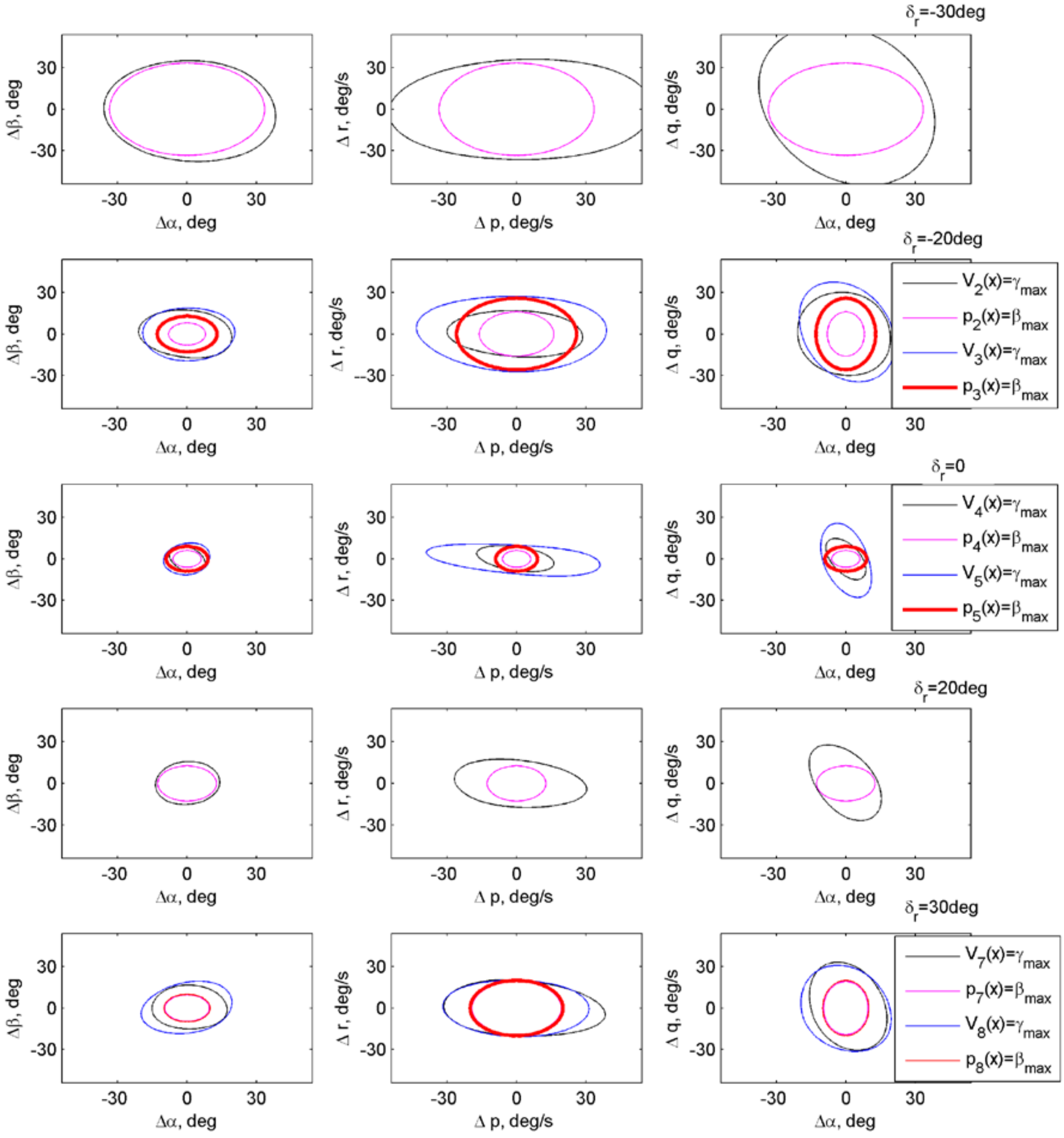

Figure 12: Projections of sublevel sets of Lyapunov functions and estimated ellipsoidal approximations for spin regimes from Table1

5. Conclusions

Recently developed quantitative methods for analysis of nonlinear systems used so far for robustness analysis of normal flight regimes has been applied to analysis of dangerous flight regimes such as spins. The parameters of spin modes of a general aircraft were calculated using numerical continuation on rudder deflection as a parameter. Then multivariable polynomial approximations of the reduced 5th order system near a number of stable spin modes were calculated and validated by simulating from different initial conditions. Lower bounds on the regions of attraction for a set of stable spin equilibrium were estimated using SOS optimization, which allows to establish guaranteed

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stability regions for the nonlinear systems. It is important to note that the ROA analysis accounts for significant nonlinearities in the aircraft dynamics. The size of regions of attractions can be considered as a metric for estimation of the relative danger of different spin modes, and as a probability of entering in a particular spin mode in the case of several stable spin modes for the same set of parameter values.

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