DOI: 10.13009/EUCASS2017-151

Optimal bi-impulse orbital transfer between coplanar orbits

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Abstract

In this paper optimal bi-impulse orbital transfer between coplanar elliptical orbits has been considered. First, an effective algorithm has been developed to find the global solution. In this algorithm, we have three independent parameters. These parameters are i) the angular position of the point on the initial orbit that the first impulse is applied, ii) the angular position of the point on the final orbit that the second impulse is applied and, iii) the semi-latus rectum of the transfer orbit. The other needed parameters can be obtained from these three parameters. By mean of our algorithm, we obtain a rapid and meticulous global solution for any arbitrary elliptical orbits. Actually, we employed the algorithm to solve a wide set of numerical examples, including co-axial and non-co-axial, similar and different shape and/or energy for the initial and final orbit. This wide set of examples allowed us to disentangle the roles of each orbital parameter, such as eccentricity, energy, and argument of periapsis, in the evaluation of the optimal transfer.

Nomenclature

a	=	semi-major axis, km
e	=	eccentricity
p	=	semi-latus rectum, km
ΔV	=	impulsive velocity change, km/s
ω	=	argument of periapsis,°
ε	=	specific energy of orbit, km^2/s^2
μ	=	Standard gravitational parameter, km^3/s^2
θ	=	Angular position on the orbital plane, °
δ	=	step size reduce coefficient
r	=	Radial distance to the center of attraction, km
V_a	=	Velocity at apoapsis, km/s
V_p	=	Velocity at periapsis, km/s
α	=	Scale factor
γ	=	Flight path angle, °
Δ	=	Initial step size
β	=	Expansion factor

1. Introduction

The orbital transfer is one of the most important and costly parts of any space mission. The cost of payload transfer to the final orbit is relatively high, because for ΔV on the order of 1 km/s or greater so that the required propellant exceeds 25 percent of the spacecraft mass prior to the burn [1]. Hence, the problem of fuel consumption optimization in orbital transfers has a long history, and many scientists have been working on this problem for many years and have worked out solutions to many special and/or simple cases [2,3]. In the preliminary satellite transfer mission design, the number of the impulses should be determined. In [4] the optimal number of impulses to transfer among circular orbits has been studied.
The optimization methods have been used to find the solutions to the orbital transfer problem are generally classified to "direct" and "indirect". Indirect solutions are those using the analytical necessary conditions from the calculus of variations [5], i.e. all the first-order partial derivative of the ΔV function with respect to independent variables should be zero. In [6], Lawden derived the equations for the optimal impulsive transfer of coplanar elliptical orbits. In [7],

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Plimmer showed solutions to the special cases of transfer between circular orbits, between an elliptical orbit and a circular orbit, between co-axial elliptical orbits, and identical (the same size and shape) non-coaxial elliptical orbits. In [8], Lawden studied the bi-impulse optimal transfer of coplanar elliptical orbits and gave the solution to a numerical example. In [9], Lawden applied the calculus of variations to the impulsive transfer problem and the necessary conditions for optimality as elements of the "primer vector" are given. Later on, in [10,11] a geometric approach to the problem is introduced which simplifies the structure of the primer vector's problem. In [12,13], the solution of the coplanar two-impulse tangential transfer is presented in closed-form. To solve the problem with a complex geometry, such as the case of optimal impulsive orbital transfer among coplanar non co-axial orbits, a system of nonlinear equations extracted from the indirect optimization method should be solved. The numerical solution of this system is very sensitive to the initial guess values, and the convergence process is usually slow. Also, solving these equations satisfies the necessary condition and lead only to one of the numerous local minima. For instance, in [14], six local minima are found for a numerical example and in [15], local minima of the three-impulse orbital transfers are presented. These adverse circumstances have motivated the development of effective "direct" methods during the course of past decades. Direct solutions transcribe the continuous optimal control problem into a parameter optimization problem. Satisfaction of the system equations is accomplished by integrating them stepwise using either implicit or explicit rules; in either case, the effect is to generate nonlinear constraint equations that must be satisfied by the parameters, which are the discrete representations of the state and control time histories. The problem is thus converted into a nonlinear programming problem [5]. Different direct methods such as Genetic Algorithms (GAs), Differential Evolutional Algorithms (DEAs), and Particle Swarm Optimization (PSO) have been used to solve the problem. In Ref. [16], the PSO is applied to impulsive orbital transfers and the problem has been solved for coplanar, non-coaxial, elliptic orbits. Some references [17-21], are made claiming the superiority of PSO over GAs and DEAs.
Overall, the literature in this field contains only solutions to a limited number of example cases. In this paper, using the presented algorithm in [22], we simulated different enormous cases and their numerical results are presented. It is an important step to analysis the behavior of the ΔV function with respect to orbital parameters of the terminal orbits.

2. Problem statement

Consider a satellite to transfer from an initial orbit to a final orbit as defined by given orbital elements. Assume that both orbits are coplanar and they are defined by the semi-major axis, eccentricity, and argument of periapsis. Assume that the satellite is subject only to the Newtonian gravitational force field of a fixed point mass body. To change the orbit of the spacecraft by impulsive maneuvers, its velocity vector should be changed once or several times using the onboard rocket engine. The total changes in the velocity vector ΔV are related to Δm, the mass of propellant consumed directly [1]. Therefore, minimizing ΔV correspond to minimization of the required propellant. Therefore, the problem is optimizing the bi-impulse transfer from the initial orbit to the final orbit and the simulation of ΔV_{opt} vs initial and final orbital parameters a_i , e_i , a_f , e_f and $\Delta\omega$.

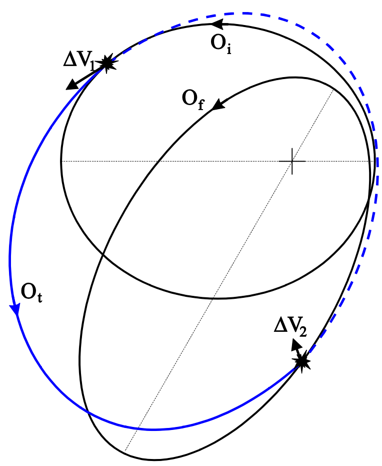

Figure 1: Schematic of optimal transfer

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Therefore, the cost function is simply

$$J = \Delta V_1 + \Delta V_2 \tag{1}$$

Regardless of the ΔV directions.
The set of relations needed to obtain our objective function ΔV are:

$$r_1 = \frac{p_i}{1+e_i\cos(\theta_1-\omega_i)}\ ,\ r_2 = \frac{p_f}{1+e_f\cos(\theta_2-\omega_f)} \tag{2}$$

$$\omega_t = \arctan\left(\frac{\frac{p_t-r_1}{r_1}\cos(\theta_2)-\frac{p_t-r_2}{r_2}\cos(\theta_1)}{\frac{p_t-r_2}{r_2}\sin(\theta_1)-\frac{p_t-r_1}{r_1}\sin(\theta_2)}\right)\ ,\ e_t = \frac{\frac{p_t-r_1}{r_1}}{\cos(\theta_1-\omega_t)}\ or\ \frac{\frac{p_t-r_2}{r_2}}{\cos(\theta_2-\omega_t)} \tag{3}$$

After obtaining the argument of periapsis and eccentricity of transfer orbit, the velocities can be obtained from vis-viva equation easily. Then, the flight path angles in the conjunction points are calculated by:

$$\begin{aligned}\gamma_{1,i} &= \arctan\left(\frac{e_i\sin(\theta_1-\omega_i)}{1+e_i\cos(\theta_1-\omega_i)}\right) & \gamma_{2,\mathrm{t}} &= \arctan\left(\frac{e_t\sin(\theta_2-\omega_t)}{1+e_t\cos(\theta_2-\omega_t)}\right)\\ \gamma_{1,\mathrm{t}} &= \arctan\left(\frac{e_t\sin(\theta_1-\omega_t)}{1+e_t\cos(\theta_1-\omega_t)}\right) & \gamma_{2,f} &= \arctan\left(\frac{e_f\sin(\theta_2-\omega_f)}{1+e_f\cos(\theta_2-\omega_f)}\right)\end{aligned} \tag{4}$$

Therefore, the ΔVs can be calculated using the law of cosines as:

$$\begin{aligned}\Delta V_1 &= \sqrt{{V_{1,i}}^2+{V_{1,t}}^2-2V_{1,i}V_{1,t}\cos(\gamma_{1,i}-\gamma_{1,t})}\\ \Delta V_2 &= \sqrt{{V_{2,f}}^2+{V_{2,t}}^2-2V_{2,f}V_{2,t}\cos(\gamma_{2,f}-\gamma_{2,t})}\end{aligned} \tag{5}$$

In the above equations, the initial, final and transfer orbital parameters are subscripted with *"i"*, *"f"* and *"t"*. Subscript "1" and "2" refers to first and second impulse respectively.

3. The Algorithm

The algorithm that we use to solve the problem is presented in [22] comprehensively. A short review of the algorithm is given in this section. It is mesh-dependent derivative-free and does not require any initial guess. Actually, the algorithm chose the best set of variables in each step and made a mesh with smaller step size in its expanded neighborhood. Let assume $x_1,x_2,...,x_n$ are the n unknown variables of the objective function $F(x_1,x_2,...,x_n)$ to be minimized. If these variables are constrained with upper and lower bounds as:

$$a_j \le x_j \le b_j \qquad (\mathrm{j}=1,2,...,n) \tag{6}$$

The algorithm works as follows:
1- Define the initial step size for every independent variable: $\Delta_j = (b_j - a_j)/N$, j=1,2,...,n
2- Define the vector for every variable: $x_j=(a_j,\ a_j+\Delta_j,\ \dots,\ a_j+N_j\Delta_j)$, j=1,2,...,n
3- Evaluate the objective function: $F(x)_{N1\times N2\times...\times Nn}$
4- Find the current solution that leads to the minimum of the objective function: $x_{current,j}, F_{min}$
5- Define the expansion factor β and step size reducer δ. They expand and refine the mesh neighborhood $x_{current,j}$.
6- Define a new vector of variables in the neighborhood of the $x_{current,j}$:$x_j=x_{cur,j}-\beta_j\Delta_j:\delta_j\Delta_j:x_{cur,j}+\beta_j\Delta_j$
7- Repeat the steps (3) to (6) till the tolerances for variables meet the desired tolerance.

We chose three independent variables (x_1, x_2, x_3), namely, the angular position of the first impulse θ_1, the angular position of the second impulse θ_2 and the semi-latus rectum of the transfer orbit p, as explained above. The algorithm needs upper and lower bounds for the independent variables. The angular positions θ_1 and θ_2 are bounded by 0 and 2π. We need to impose upper and lower bounds to p. According to our experiments, the limitation p by $0 \le p \le 2\times max(a_i,a_j)$ is satisfactory. Initial steps $\Delta\theta_1$, $\Delta\theta_2$ and $\Delta p$are set to 10°, 10°, and100km respectively. The expansion factor and step size reducer in our algorithm are set to 10 and 0.25 respectively. It means that in each loop, the algorithm expands the

DOI: 10.13009/EUCASS2017-151

mesh 10 times wider and 4 times more accurate in the neighborhood of the current variables. In each loop, finding the solution inside the current mesh guarantees the convergence of the algorithm. Otherwise, the expansion factor should be increased. The desired tolerances of the variables are set 0.01° for $\Delta\theta_1$ and $\Delta\theta_2$ and 0.01km for p.

4 Preliminary Results

In this section, the ΔV is calculated for different cases and finally, the ΔV plots with respect to initial and final orbital parameters are demonstrated. In the simulation, the semi-major axis of the initial orbit is equal to 10,000 km. In the figures 2 to 5, the ΔV is plotted vs $a_f/a_i \geq 1$. Indeed, $a_f/a_i = 1$ refers to equal energy initial and final orbits and this case has been studied in [22] in detail and an accurate fitting function has been presented in that study. To find the optimal ΔV transferring from other initial orbits with the different semi-major axis or the cases $a_f/a_i < 1$, the obtained result in this paper should be proportioned with an appropriate coefficient. Finding the coefficient is simple and it can be obtained as $\alpha = a_i/10{,}000$ and the ΔV_{opt} should be proportioned with coefficient $1/\alpha^{0.5}$. In figures 2 to 5, the lower curve refers to $\Delta\omega = 0°$ and the upper curve refers to $\Delta\omega = 180°$. Therefore the ΔV_{opt} is always between ΔV_{opt} when the terminal orbits are coaxial and their periapsides are in same direction and ΔV_{opt} when the terminal orbits are coaxial and their periapsides are in opposite direction regardless of the amount of their eccentricity and energy.

In all of the curves, there is a local minimum for ΔV_{opt} with respect to a_f/a_i. This minimum refers to the case that the final orbit tangent the Initial orbit, i.e. the radial distant (r) and its partial derivative of angular position ($\partial r/\partial \theta$) on final orbit be equal to the amount on initial orbit, respectively. This is a nonlinear system of equation and there are two answers for θ and a. One refers to inner tangent orbit and the other one refers to outer tangent orbit (Figure 6). The local minimum in the curves refers the outer tangent orbit where $a_f/a_i \geq 1$.

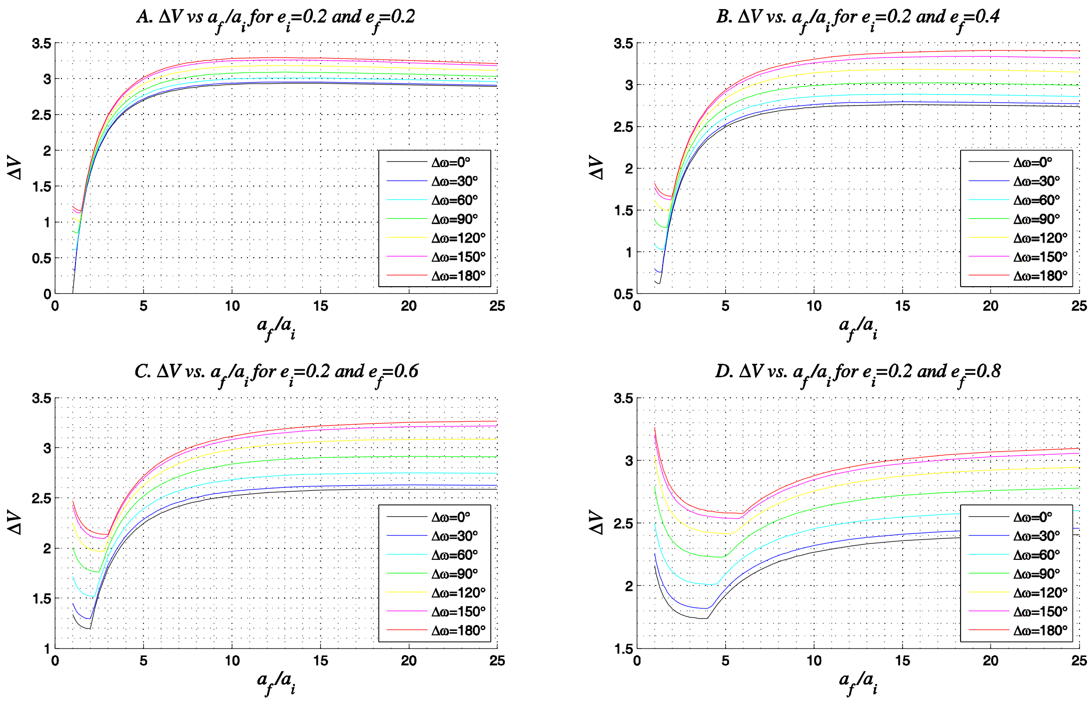

Figure 2: ΔV_{opt} vs. a_f/a_i when e_i=0.2 for various e_f and $\Delta\omega$

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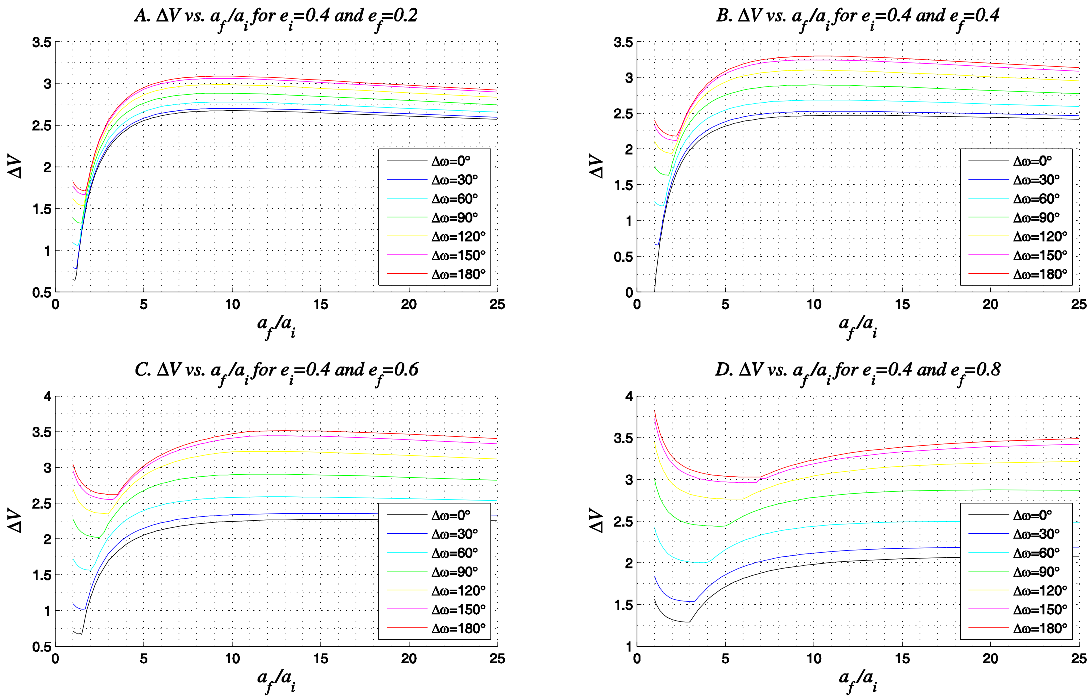

Figure 3: ΔV_{opt} vs. a_f/a_i when e_i=0.4 for various e_f and Δω

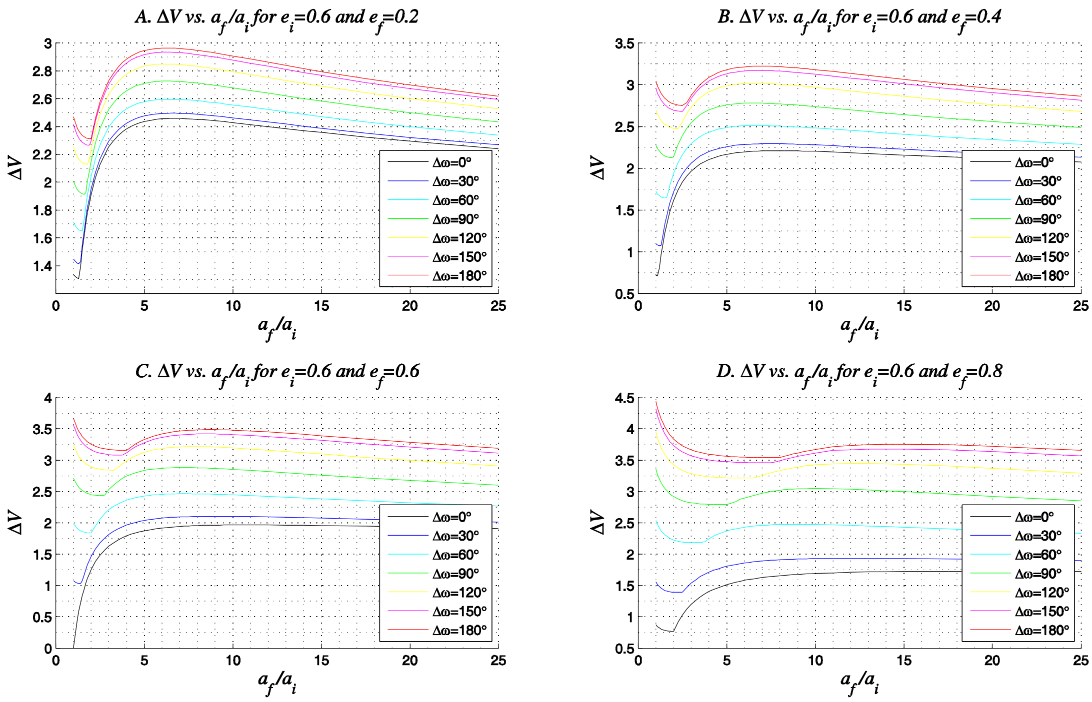

Figure 4: ΔV_{opt} vs. a_f/a_i when e_i=0.6 for various e_f and Δω

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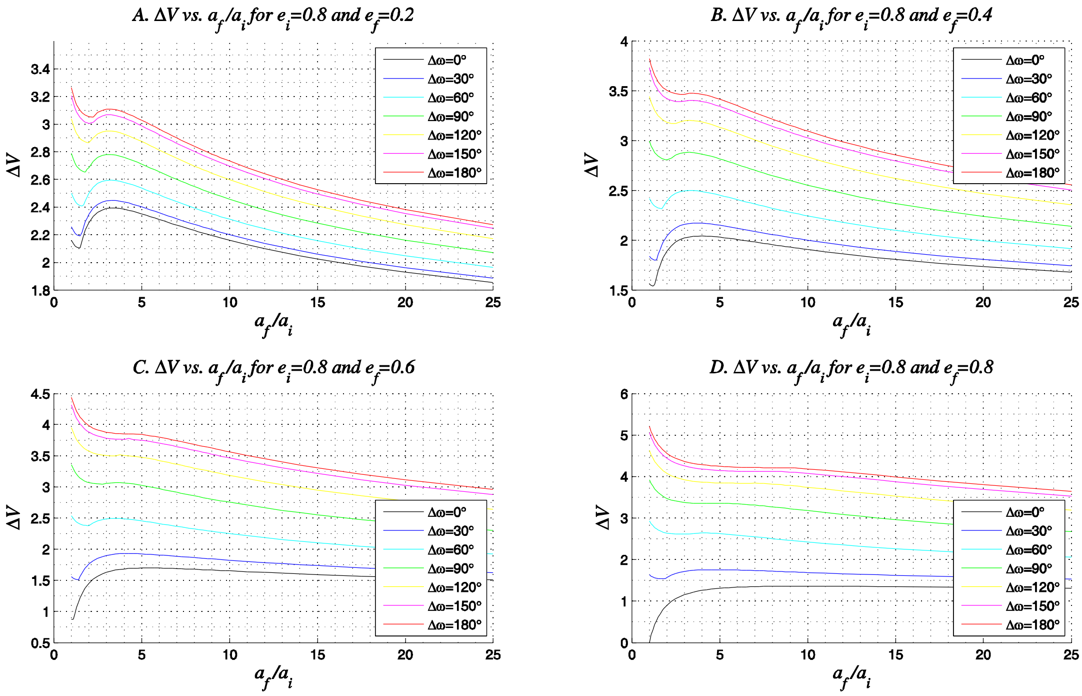

Figure 5: ΔV_{opt} vs. a_f/a_i when e_i=0.8 for various e_f and Δω

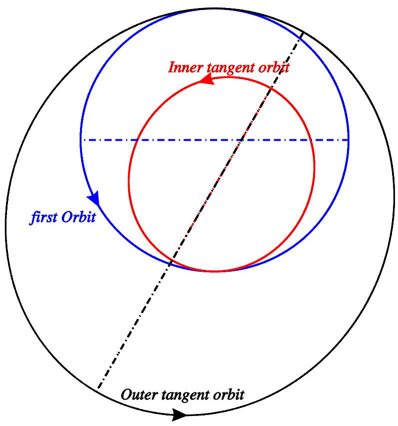

Figure 6: Schematic of inner tangent and outer tangent orbits. The eccentricity and argument of periapsis of these orbits are the desired eccentricity and argument of periapsis and they are tangent to initial orbits

5 Conclusion

In this paper, we dealt with the problem of determination the minimum required velocity impulse to make an orbital transfer between coplanar terminal orbits. First, we introduced an algorithm that attains the global solution for any arbitrary coplanar elliptical orbits, in a quick and precise way. Second, a large set of numerical examples were simulated. The numerical results described the behavior of the ΔV function with respect to orbital parameters of the terminal orbits e_i, a_f/a_i, e_f, and Δω meticulously. The analysis of ΔV function may lead to an accurate fitting function to the result as a function of the various input parameters, to provide simple formulas for a quick evaluation of the minimum velocity impulse for the transfer.

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