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Problems with discontinuous boundary conditions describing laminar flows at high Reynolds numbers

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Introduction

Discontinuous boundary conditions in the theory of the boundary layer are found in the problems, describing the flow near surface distortions, near the edges of wings. There are some methods of boundary layer flow control leading to the abrupt change in boundary conditions , for example, tangential blowing, suction, nonzero surface velocity, etc. The impact of the surface motion layer has been demonstrated by Prandtl in the experiment with a rotating cylinder in 1904. The monograph [Schlichting G., 1968] describes experiments of Favre, which found that the profile with a partially movable upper surface is maintained unseparated to the angle of attack of 55 °. In [Zhuk VI, OS Ryzhov, 1979; Zhuk VI, 1980; LA Sokolov, 1980] studied the interaction of moving at a constant velocity of the shock wave from the laminar boundary layer and was shown that in some cases disturbed can be described by a system of equations for steady state "free interaction" with a non-zero surface speed. With the sudden beginning or ending of the movement surface gap in the boundary conditions perturbs the flow in the boundary layer. The classical theory of the boundary layer may not be valid for the description of such flows.

The basis of the analysis of problems with suddenly changing boundary conditions may provide method of matched asymptotic expansions, essentially for the first time applied in formulating the Prandtl boundary-layer theory and the method of coordinate expansions used in [Goldstein S., 1931] for the flow study in the wake of the plate of finite length.

The flow analysis near the flat plate of finite length based on the asymptotic analysis of the Navier-Stokes equations showed [Messiter A.F., 1970; Stewartson K. 1970] showed that the vertical velocity at the outer edge of the viscous flow due to changes in the displacement thickness is limited in its growth and does not exceed values at which the induced shear flow in the external pressure disturbance begins to affect a change in the displacement thickness. Similar effects of local strong viscous-inviscid interaction were found in the vicinity of the point of separation of the laminar from a smooth surface in a supersonic flow [Neyland VJ, 1969a; Stewartson K., Williams P.G., 1969]. Further analysis [Veldman A.E.P., 1976] showed that near the trailing edge of the plate there is a complex flow structure, which includes a number of sub-areas within which the flow is

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described by the full Navier-Stokes equations, by system of boundary layer equations with induced pressure gradient, and others.

Problem formulation

As an example, we will consider the problem of the flow near the velocity discontinuity region on a flat plate having a region moving at a velocity u_w at a distance from the leading edge (Fig. 1). The same designations as in the preceding sections are taken for

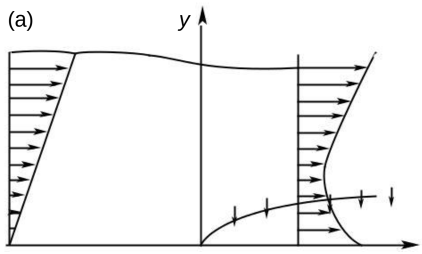

Fig.1

the Cartesian coordinates measured along the surface and normal to it: time, the velocity components, the density, the viscosity, and the total enthalpy.

Let us consider the structure of the disturbed steady flow for which $u(x < 1) = 0$ and $u(x > 0) = u_w > 0$. The difference in the velocities of streamlines flowing near the surface at a velocity u_w for $x > 1$ and those with near-zero velocities for $x < 1$ can lead to the formation of a new boundary layer downstream of the point of discontinuity in the boundary condition.

$$\varepsilon = \mathrm{Re}^{-1/2},\ \mathrm{Re} = \rho_\infty u_\infty l / \mu_\infty \qquad (1)$$

$$y \sim \varepsilon u_w^{-1/2} x^{1/2} \qquad (2)$$

The estimate for the thickness of the newborn boundary layer follows from the condition of the equality of the orders of the terms that describe the effects of the inertia and viscosity forces in the longitudinal momentum equation

At a fixed surface velocity and a variable thickness of the newborn boundary layer the friction in the latter decreases monotonically with increasing longitudinal coordinate. Using estimate (2) we can determine the distance x_1 at which the friction in the new boundary layer becomes comparable with that in the main boundary layer

$$u_w / \varepsilon \sim 1/ \varepsilon,\ x_1 \sim u_w^3 \qquad (3)$$

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This formula can be conveniently presented in the form of the dependence ln x_i/ln $\varepsilon = f$ (ln u_w/ln ε); then eq. (2) is presented as line *OB* in Fig. 2

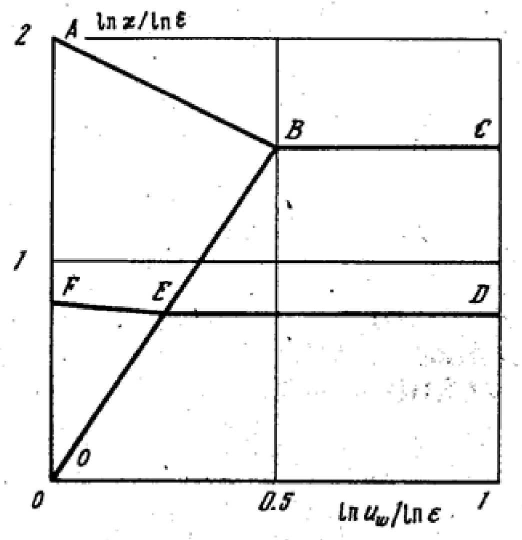

Fig.2

From estimate (2) we also can determine the distance x_2 from the point of the discontinuity in the boundary condition, at which the nonlinear disturbance region thickness and length become of the same order and where, in essence, the assumptions of boundary layer theory are violated

$$x_2 \sim \varepsilon^2 / u_w \qquad (4)$$

Equation (4) is presented by line *AB* in Fig. 2

Then the coincidence of the longitudinal and transverse scales leads to the equality of the orders of the disturbed longitudinal and transverse velocities. Since Eq. (2) was obtained under assumption of the equality of the orders of the inertia and viscosity forces, it can be shown that the flow in a region with the scales $x_2 \sim y_2 \sim \varepsilon^2 / u_w$ is described by the complete system of incompressible Navier–Stokes equations. An analogous region appears also in considering the flow in the vicinity of the leading edge of a zero-thickness flat plate.

Let us estimate the effect of the boundary layer formed as a result of the discontinuity in the boundary conditions on the flow in the main boundary layer formed near the fixed plate (since at the bottom of the main boundary layer a new boundary layer is formed and these layers need to be distinguished, in what follows we will call them the main and newborn boundary layers). Physically, this effect manifests itself as gas absorption from

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the main boundary layer. The estimate for the vertical velocity in the newborn boundary layer follows from Eq. (3.91) and the continuity equation and takes the form:

$$v \sim \varepsilon u_w^{1/2} x^{-1/2}, \quad \psi \sim vx \sim \varepsilon u_w^{1/2} x^{1/2} \tag{5}$$

Absorption of this flow rate from the original boundary layer over the length x leads to a variation of its thickness. For determining this variation we will use the representation $u \sim y/\varepsilon$ of the velocity profile in the main boundary layer at small distances as compared with the boundary layer thickness. Accordingly, at the distance y from the surface the gas

flow rate through the main boundary layer can be estimated as follows: $\psi \sim y^2/\varepsilon$. Therefore, the estimate of the variation of the boundary layer displacement thickness takes the form:

$$\Delta\delta \sim \varepsilon^{1/2} \psi^{1/2} \sim \varepsilon u_w^{1/4} x^{1/4} \tag{6}$$

This variation of the displacement thickness induces the corresponding pressure variation in the outer inviscid flow

$$\Delta p \sim \Delta\delta / x \sim \varepsilon u_w^{1/4} x^{-3/4} \tag{7}$$

The latter estimate follows from the linear theory of inviscid (both subsonic and supersonic) flows. Using this theory is justified if the distance x_3, over which the above-mentioned effects are important, is greater in the order than the main boundary layer thickness $\boldsymbol{\delta \sim \varepsilon}$. The fulfillment of the condition $x_3 > O(\varepsilon)$ can be verified if the estimate of the distance over which the interaction effects manifest themselves is obtained.

The estimate for the pressure disturbance makes it possible to determine the distance x_3 over which the induced pressure gradient has a nonlinear effect on the wall region of the main boundary layer. For further analysis it is important to note that a gas flow rate is absorbed from precisely this region and the variation of the thickness of precisely this region determines the total variation of the boundary layer displacement thickness

$$\Delta p \sim u_3^2, \quad x_3 \sim \varepsilon^{4/5} u_w^{-1/5} \tag{8}$$

The second relation (8) is presented by line *EF* in Fig. 2. Using estimate (8) we can write the condition at which the interaction region length is greater than the boundary layer thickness

$$u_w < 1/\varepsilon$$

This inequality is assumed to be necessarily fulfilled.

The characteristic points *B* and *E* are at intersections of line *OB* with lines *AB* and *FE*. Typical for flow regimes corresponding to points *B* and *E* is the coincidence of the orders of the friction in the disturbed zone and the main boundary layer. Then point *B* is associated with the flow described by the system of Navier–Stokes equations and point *E* with the flow described by the system of equations of free interaction theory.

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Estimate (8) is invalid for the region of variation of the parameter u_w located to the right of point *B* and in the cases in which the linear disturbance regime is realized due to a greater relative viscosity effect. The equality of the orders of the terms describing the effects of the viscosity and inertia forces leads to the estimate

$$x_4 \sim \varepsilon^{3/2} \qquad (9)$$

Relation (9) is presented by line *BC* in Fig. 2. Similar considerations can be used for determining the distance x_5 over which linear processes of viscous–inviscid interaction occur. Relation (9) is invalid in the region of variation of the parameter x_5 located to the right of point *E*. For this distance the following estimate holds

$$x_5 \sim \varepsilon^{3/4} \qquad (10)$$

which is associated with line *ED*.

The diagram of the disturbed flow regions plotted in Fig. 2 makes it possible to determine the dimensions of these regions and the nature of the corresponding flows for a given amplitude of the parameter u_w. Thus, the effect of the disturbance with an amplitude $O(\varepsilon^{1/4}) \le u_w \le O(1)$ consists in the appearance near the discontinuity of a region with dimensions determined by line *AR*, where the flow is described by the system of incompress-ible Navier–Stokes equations. Next in extent is the region whose longitudinal dimension is determined by line *EF*, where the flow in the first approximation is described by the Burgers equation. At intermediate distances, with variation of the parameters in the region between lines *AB* and *EF* the viscosity effect is inessential in the flow in the nonlinear disturbance region and the compensation interaction is realized. The absence of viscous terms from the equations governing the disturbed flow requires introducing a subdomain in which the viscosity and inertia forces are of the same order. At the same time, there exists a region with a length determined by line *OB* in which the viscosity effect is essential and the surface friction is of the same order as that in the original boundary layer. As noted above, point *E* corresponds to the general case in which nonlinear processes of equalization of the friction of interaction with the outer flow occur in the same region, namely, the free interaction region (Neiland, 1969*a*; Stewartson and Williams, 1969).

When the parameter u_w varies on the range $O(\varepsilon^{1/4}) \le u_w \le O(1)$, apart from the above-mentioned region, in which the flow is described by the system of Navier–Stokes equations, there appears one more region whose length is determined by line *BE*. Here, the flow is described by the system of boundary layer equations with a compensation condition of interaction. In this region, the surface friction is equalized. Finally, for $u_w \sim \varepsilon^{1/2}$ friction is equalized directly in the region in which the flow is described by the system of Navier–Stokes equations.

Thus, near the discontinuity point (line) there occurs a system of embedded regions with different longitudinal scales. For given u_w the dimension of each of these regions can be determined using the diagram in Fig. 2. It should also be borne in mind that each region, whose longitudinal dimension is greater in the order than the boundary layer thickness, consists of subdomains with different transverse dimensions.

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Using the diagram in Fig. 2 and the above estimates we can determine the conditions at which time-dependent effects can manifest themselves in the disturbed flow regions considered. For this purpose we will determine the characteristic temporal scales equal to the ratios of the region lengths to the corresponding characteristic velocities. Thus, for the region under consideration, consisting of a system of embedded subdomains, the greatest characteristic time is associated with the subdomain with the least characteristic longitudinal velocity, while time-dependent processes in the subdomain with the greatest scale time are associated with quasi-stationary processes in the other subdomains. As follows from the estimates presented above, the least longitudinal velocity is characteristic of the region in which nonlinear variations take place.

It should also be taken into account that the regions with different longitudinal dimensions are associated with different characteristic times. Thus, for the regions corresponding to the lines presented in Fig. 2 we have the following estimates: $t_2 \sim \varepsilon^2 u_w^{-2}$ at *AB*, $t_3 \sim \varepsilon^{3/5} u_w^{-2/5}$ at *EF*, and $t_1 \sim u_w^2$ at *OB*.

For further analysis we note that for $u_w = O(\varepsilon^{1/2})$ the least time is characteristic of the flow region corresponding to line *AB*, then for $u_w > O(\varepsilon^{1/2})$ the next in duration is time for region *EF*, and the greatest characteristic time is that for the region corresponding to line *OE*.

Analysis of the regimes described by free interaction theory

For $u_w \sim \varepsilon^{1/4}$ the flow in the region with nonlinear variations of the flow functions is described by the system of equations for the free interaction regime. This system of equations is as follows:

$$\Psi'\dot{\Psi}' - \dot{\Psi}\Psi'' + \dot{P} = \Psi'''$$

$$\dot{\Psi}'(X,\infty) = -P, \quad \Psi(X,0) = 0, \quad \Psi(-\infty, \mathrm{Y}) = \mathrm{Y}^2/2 \qquad (11)$$

$$\Psi'(X<0,0) = 0, \quad \Psi'(X>0,0) = U_w$$

where next similarity variables are introduced

$$X = (a^5\beta^3\rho_w^2\mu_w\varepsilon^{-3})^{1/4}x; \quad Y = (a^3\beta\rho_w^2\mu_w^{-1}\varepsilon^{-5})^{1/4}y \qquad (12)$$

$$\Psi = (a\beta\rho_w^2\mu_w^{-1}\varepsilon^{-3})^{1/2}\psi; \;\; P = (a^{-1}\beta\mu_w^{-1}\varepsilon^{1})^{1/2}p;$$

$$U_w = (a^{-1}\beta\rho_w^2\mu_w^{-1}\varepsilon^{-1})^{1/4}u_w; \;\; \beta = (M_\infty^2 - 1)^{1/2}$$

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$$a = \varepsilon \frac{\partial u}{\partial y}(X \to -\infty, 0)$$

For U_w <<1 the solution of the boundary value problem (12) can be represented in the form:

$$\Psi = Y^2/2 + f(X,Y)U_w: \ \ P = U_w \overline{P} \qquad (13)$$

$$Y\dot{f}' - \dot{f} + \dot{\overline{P}} = f'''$$

$$f'(X,\infty) = -\overline{P}; \quad f'(X<0,0) = 0; \quad f'(X>0,0) = 1$$

$$f(X,0) = 0; \quad f(-\infty,Y) = 0$$

Using the Fourier transformation the solution of the boundary value problem (13) can be obtained in the form:

$$P(X<0) = -3U_w\theta \exp(\theta X)/4; \ \ \theta = [-3Ai'(0)]^{3/4} \qquad (14)$$

$$P(X>0) = -\frac{3^{1/2}Uw\theta}{2\pi}\int_0^\infty \frac{s^{4/3}\exp(-\theta s X)ds}{1+s^{4/3}+s^{8/3}};$$

The disturbed flow near the point, at which the motion of the surface stops, can be similarly described. It is governed by the system of equations (13) in which the boundary conditions for the function (X, 0) take the form:

$$\Psi'(X<0,0) = U_w$$

$$\Psi'(X>0,0) = 0$$

The solution of the corresponding linear boundary value problem for $U_w <<1$

$$P(X<0) = 3U_w\theta \exp(\theta X)/4; \qquad (15)$$

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$$P(X>0)=\frac{3^{1/2}U_w\theta}{2\pi}\int_0^{\infty}\frac{s^{4/3}\exp(-\theta s X)ds}{1+s^{4/3}+s^{8/3}}; \quad (16)$$

It should be noted that for U_w <<1 the solutions governing the flows near the points of the beginning and cessation of the motion of the surface directed counter to the undisturbed oncoming flow differ from solutions (15) and (16) only in sign.

In the vicinity of point X = +0 for Y = O(1) the flow functions can be represented in the form of the following coordinate expansions

$$\Psi=\Psi_0(X=-0,Y)+X^{1/2}\Psi_1(Y)+...;$$

$$P(X)=P_0(X=-0)+X^{1/2}P_{10}$$

The functional form of the expansions is determined from the conditions of the matching with the solution in the newborn boundary layer, that is, for Y = O(1), where the flow functions can be represented in the form:

$$\Psi=2^{1/2}X^{1/2}U_w^{1/2}f(\eta)+...;\quad \eta=2^{-1/2}YU_w^{1/2}X^{-1/2}$$

Substituting Eq. (3.102) in the system of equations (3.97) leads to the following equation for the function $_1(Y)$:

$$\Psi_0'\Psi_1'-\Psi_1\Psi_0''+P_1=0$$

An analysis of expressions (3.102) and (3.103) and the interaction conditions shows that for nonzero values of the function $_1(Y\to\infty)$ nonzero disturbances imposed on the outer boundary of the region with nonlinear flow function variation lead to infinitely large negative values of the induced pressure; therefore, condition $\Psi_1(Y\to\infty)$ must be fulfilled. Thus, a rapid decrease or increase in the displacement thickness at the cost of the newborn boundary layer must be accompanied by the appearance of a large pressure gradient ensuring zero, in the leading term, total variation of the displacement thickness. The solution for the function $\Psi_1(Y)$ takes the form:

$$\Psi_1=-\Psi_0'P_{10}\int_Y^{\infty}\frac{dY}{\Psi_0'^2}$$

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Matching with the solution for the newborn boundary layer makes it possible to determine the parameter P_{10}; in the case of the beginning of the motion of the surface it is as follows:

$$P_{10} = 2^{1/2} C_0 a U_w^{1/2}; \quad C_0 \approx 1.229$$

Or

$$P_{10} = -2^{1/2} C_1 a U_w^{-1/2} J^{-1}; \quad C_1 \approx 1.217; \quad J = \int_0^\infty \frac{dY}{\Psi_0'^2}$$

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