DOI: 10.13009/EUCASS2017-82

New type of singularity in the near-wall region of 3D boundary layer over the runoff plane and the flow structure in its vicinity

V.I. Shalaev
Moscow Inctitute of Physics and Technology (State Univercity)
Central Aerohydrodynamic Institute
Russia, Zhukovsky, 140180

Abstract
Singularities appearing in solutions of 3D boundary layer (BL) equations are discussed. For conical bodies, equations are investigated analytically using asymptotic methods. Explicit solutions are obtained for the outer BL region; their singularities are studied. The asymptotic flow structure near the singularity is constructed on the base of Navier-Stokes equations at large Reynolds numbers. For different flow regions analytical solutions are found and are matched with BL equation solutions. Properties of BL equations for the near-wall region in the runoff plane are investigated and a criterion of the solution disappearing is found. It is shown that this criterion separates two different topological flow structures and corresponds to the singularity appearance in this plane in solutions of full equations. Calculations confirmed these results are presented. These results are important to understand properties of different aerodynamic flows with regions, where two streamline families collide between themselves, and to construct effective numerical methods.

1. Introduction

Singular solutions of unsteady or 3D boundary layer (BL) equations are slightly studied due to difficulties of analytical investigations of complex nonlinear equations and the uncertainty of numerical results treatments. However, this task will be solved, since it is of interest as for the mathematical physics and for numerical modelling of aerodynamic applications at high flight velocities. In contrast to the 2D steady BL equations, considered singularities don’t related directly with the flow separation, however their study is necessary to understand this phenomenon in the 3D flow and, apparently, different separation development scenario may be at the presence of different singularity types.

For the first time, a singularity was found in the solution of unsteady BL equations by K. Stewartson for the flow around the flat plate impulsively set into motion [1]. It had the logarithmic type and located in the outer BL part on the boundary of the unsteady flow region. The singularity of the similar type was discovered on the side edge of a quarter flat plate in a uniform freestream [2] and at a collision of two jets [3, 4]. In the reference [5], necessary conditions were formulated for a singularity formation in self-similar unsteady and 3D boundary layers. However as sufficient conditions and singularity types were not obtained.

It can understand from these works, the singularity can arise when two subcharacteristic (streamlines) families are collided – this is necessary condition. Such situation arises usually in the leeward symmetry (runoff) plane over a body of revolution under the angle of attack α^*. Unusual properties in numerical solutions of self-similar equations in this plane for a round slender cone in supersonic freestreams were studied in many works due to the practical interest of the heat exchange on flying vehicles head parts [6 – 12]. In this case, the one parameter defines the flow, $k = 4\alpha^*/(3\delta_c)$, where δ_c is cone half apex angle. Two solutions were found in the windward symmetry (attachment) plane and at small angles of attack ($k \le k_c$) in the leeward symmetry plane. In this plane, no solutions were found at moderate angles of attack ($k_c \le k \le 2/3$), and many solutions at larger incidences up to BL separation ($2/3 \le k < 1$). Full BL equation solutions with initial conditions in the windward symmetry plane fixed the violation of symmetry conditions in the runoff plane, a velocity jump through this plane in the angle of attack diapason, when the self-similar solution been absent [13, 14]. Similar results were obtained for the turbulent flow over the wing [15].

DOI: 10.13009/EUCASS2017-82

The task for the cone was solved numerically on the base of parabolized Navier-Stokes equations, without the streamwise viscous diffusion [16]. However the problem is retained since the flow structure and reasons of unusual BL properties have not been explained.

Analytical solutions of full equations for the outer BL part on the slender round cone with initial conditions in the windward symmetry plane showed the singularity presence in the leeward symmetry plane of the logarithmic type at $k=1/3$ and of a power type at $k>1/3$ [17]. It had been showed that full equations numerical solutions gave incorrect results near the singularity due to the accuracy loss. Similar but more complex results were obtained for arbitrary cones; they allow defining the sufficient conditions of the singularity arising [18, 19]. The asymptotic flow structure at large Reynolds number near the singularity on the base of Navier-Stokes equations was constructed, and analytical solutions in different asymptotic regions were obtained, which were matched with BL solutions. The analysis of the viscous-inviscid interaction region, in particular, revealed that the singularity can arise not only in self-similar but in full 3DBL equations The theory showed that the singularity appearance relates with eigensolutions of the BL equations appearing near the runoff plane; it also explained numerical modelling results on the base of parabolized Navier-Stokes equations.

In the outer BL part the theory gives the critical angle of attack for the singularity appearance $k_c=1/3$. However calculations showed that this parameter is a function on numbers of Mack M_∞, Prandtl Pr and the wall temperature h_w, $k_c=k_c\left(\mathrm{M}_\infty,\mathrm{Pr},h_w\right)$ [6–12]. This indicates that a singularity can arise in the near wall region, and new results presented below confirm it. The series decomposition of the near-wall solution in the runoff plane showed the presence of a parameter α, the linear combination of skin friction components, the sign change of which leads to the change of the physical flow topology near this plane. The analysis of BL equations in the near-wall region showed that $\alpha=0$ corresponds to the critical value k_c, and it confirmed by all published numerical calculations [6–12, 19–21]. In the runoff plane, the new power type singularity in solutions of full BL equations was revealed that is related with the eigensolutions appearing near this plane. Presented calculation results for BL on delta wing confirm the singularity presence.

2. Problem formulation

The 3D laminar boundary layer on a conical surface in the orthogonal coordinate system $xy\varphi$ (Fig. 1) is described by following self-similar equations and boundary conditions [18, 20]:

$$\begin{gathered}
u_{yy}=Awu_\varphi+vu_y+A_1w(u-w),\ \ w_{yy}=Aww_\varphi+vw_y+w\left(\tfrac{2}{3}u+Kw\right)-h\left(\tfrac{2}{3}+K\right),\\
h_{yy}=Awh_\varphi+vh_y-M_e\left(u_y^2+\tfrac{3}{2}A_1w_y^2\right),\ \ \rho h=1,\ \ y=\varepsilon\sqrt{\frac{3\rho_eu_e}{2x\mu_e}}\int_0^{y^*}\rho d\frac{y^*}{l},\ \ \mathrm{Re}=\varepsilon^{-2}=\frac{\rho_\infty u_\infty l}{\mu_\infty},\\
f_y=u,\ \ g_y=w,\ \ v=-f-\left[K-\tfrac{1}{2}A\left(\ln\left(\rho_e\,\mu_e/u_e\right)\right)_\varphi\right]g-Ag_\varphi,\\
y=0:u=v=w=0,\ \ h=h_w\ \ (h_y=0);\ \ y=\infty:u=w=h=1.
\end{gathered}\tag{1.1}$$

Equation coefficients are defined by expressions:

$$M_e(\varphi)=(\gamma-1)M_\infty^2\frac{u_e^2}{h_e},\ \ K(\varphi)=\frac{2w_{e\varphi}}{3Ru_e},\ \ A(\varphi)=\frac{2w_e}{3Ru_e},\ \ A_1(\varphi)=\frac{2}{3}\left(\frac{w_e}{u_e}\right)^2.$$

In these equations, to reduce formulas $\mathrm{Pr}=1$ and the linear dependence of the viscosity on the temperature ($\rho\mu=1$) are assumed. Indexes y and φ denote derivatives with respect corresponding variables; x is distance from the body nose along the generator referenced to the body length l; y is Dorodnitsyn variable; y^* is normal to the body surface; φ is transversal coordinate, it can be the polar angle for a round cone (see Figure 1); $f(y,\varphi)$ and $g(y,\varphi)$ are longitudinal and transverse streamfunctions; $v(y,\varphi)$ is transformed normal velocity; $R(\varphi)$ is metric coefficient. The density ρ, the enthalpy h, the viscosity μ, the longitudinal u and transversal w velocities are referenced to them values at the outer boundary indexed by e, which are normalized to their freestream values

DOI: 10.13009/EUCASS2017-82

indexed by ∞; they are functions of φ only. The transversal velocity on the outer boundary-layer edge $w_e = 0$ in the initial value plane (the attachment plane) $\varphi = 0$, in which $K(0) > 0$), and in the runoff plane $\varphi = \varphi_1$, in which $K(\varphi_1) = -k < 0$ and two boundary layer parts came from different sides of the attachment plane are collided. For the round cone $\varphi_1 = \pi$.

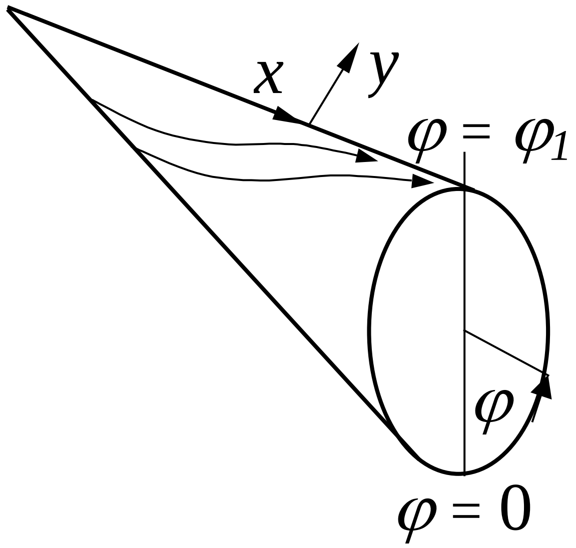


Figure 1: The general flow scheme and the coordinate system.

Eqs. (1.1) are simplified for slender bodies since in this case $A_1 << 1$, $u_e = \rho_e = \mu_e = 1$. Neglecting proportional to A_1 terms in (1.1) we obtain Crocco integral for the enthalpy and momentum equations in the form:

$$\begin{gathered} h = h_w + h_r u - \tfrac{1}{2} M_e u^2, \quad h_r = 1 - h_w + \tfrac{1}{2} M_e, \quad M_e = (\gamma - 1) M_\infty^2, \;\; v = -\left(f + Kg + Ag_\varphi\right), \\ u_{yy} = Awu_\varphi + vu_y, \;\; w_{yy} = Aww_\varphi + vw_y + w\left(\tfrac{2}{3}u + Kw\right) - h\left(\tfrac{2}{3} + K\right). \end{gathered} \tag{1.2}$$

For the slender round cone $w_e = 2\alpha^* \sin\varphi$, $K(\varphi) = k\cos\varphi$, $A(\varphi) = k\sin\varphi$.

3. Singularities in boundary layer outer part and corresponding flow structure

We consider the asymptotic form of Eqs. (1.2) at $y >> 1$, so that flow functions are represented as [18, 20]:

$$\begin{gathered} u = 1 + U(\eta, \varphi), \;\; w = 1 + W(\eta, \varphi), \;\; \eta = (y - \delta)\big/\sqrt{a(\varphi)}, \\ h = 1 + H = 1 - \left(\tfrac{1}{2} M_e + h_w - 1\right)U - \tfrac{1}{2} M_e U^2 \end{gathered} \tag{2.1}$$

Here $\delta(\varphi)$ is displacement thickness defined by the equation of F. Moore [6], the function $a(\varphi)$ is found from the local self-similarity condition, $U << 1$ and $W << 1$ are velocity perturbations with respect to boundary conditions, which in the first order approximation satisfy to equations [18, 20]:

$$U_{\eta\eta} + \eta U_\eta - aAU_\varphi = 0, \;\; W_{\eta\eta} + \eta W_\eta - \tfrac{2}{3} a\left[\tfrac{3}{2} AW_\varphi + (1 + 3K)W\right] = \tfrac{2}{3} ap(\varphi)U. \tag{2.2}$$

Equations (2.2) have solutions:

$$U(\eta, \varphi) = C_1 erfc\left(\eta\big/\sqrt{2}\right), \quad W(\eta, \varphi) = -b(\varphi)U, \quad W_1(\eta, \varphi) = -b(\varphi)U + B_1(k)V(\eta, \varphi). \tag{2.3}$$

DOI: 10.13009/EUCASS2017-82

Constants C_1 and B_1 are calculated from matching condition with a numerical solution inside the boundary layer. These solutions satisfy to initial conditions in the attachment plane and must tend to zero at $\eta \to \infty$. The function $V(\eta,\varphi)$ is the solution of the homogeneous equation for the cross velocity perturbation (2.2), when the right hand side equals to zero; it is expressed by Veber-Ermit functions [22]. The coefficient $B_1 \sim 1/K(0)$, i.e. it has the singularity at $K(0) \to 0$. For the round cone this limit corresponds to zero angle of attack; in this case, the analytical expression for $W_1(\eta,\varphi)$ shows the presence of the power type singularity in the leeward plane $\varphi = \varphi_1$ [17]. The first solution $W(\eta,\varphi)$ is regular in this limit, its behaviour is defined by functions $a(\varphi)$ and $b(\varphi)$, which satisfy to equations [17 – 21]:

$$\begin{gathered} w_e b_\varphi + 2(1+M) w_{e\varphi} b = 2pMw_{e\varphi}, \quad p(\varphi) = 1 + \left(1 + \tfrac{3}{2}K\right)\left(\tfrac{1}{2}M_e + h_w - 1\right), \\ w_e a_\varphi + 2(N+1) w_{e\varphi} a = 2Nw_{e\varphi}, \quad N(\varphi) = 3M(\varphi) = K^{-1}. \end{gathered} \tag{2.4}$$

The equation for the function $a(\varphi)$ is derived from the condition that coefficient before second terms in Eqs. (2.2) equals to unity; this corresponds to the local self-similarity of BL equations near the runoff plane. Solutions of these equations with initial conditions in the attachment plane are represented in integral forms in the general case and have analytical expressions for the round cone [17–21]. Their properties near the leeward plane, at $\zeta = \varphi_1 - \varphi << 1$, are represented by expressions [19–21]:

$$\begin{gathered} w_e = \tfrac{3}{2}kR\zeta,\ k = -K(\varphi_1),\ R = R(\varphi_1),\ p_1 = p(\varphi_1),\ n = 3m = -K^{-1}(\varphi_1), \\ m \neq 1: b = \frac{mp_1}{m-1} - b_m \zeta^{2(m-1)}, \quad m = 1: b = -2p_1 \ln\zeta + b_1, \\ n \neq 1: a = \frac{n}{n-1} + a_n \zeta^{2(n-1)}, \quad n = 1: a = -2\ln\zeta + a_1. \end{gathered} \tag{2.5}$$

Here a_n and b_m are known coefficients [17–21]. Formulas (2.5) are true for non-slender bodies also [18]. These results show the presence in the outer BL part of two singularity types in the leeward plane related with properties of functions $a(\varphi)$ and $b(\varphi)$.

For $k < 1$ the function $U(\eta,\zeta)$ exists at $\zeta = 0$ but reaches this limit irregularly, its behaviour is studied analytically in details for the slender round cone [17]. For $k \geq 1$ the function $U(\eta,\zeta)$ is singular at $\zeta \to 0$ since $a(\zeta) \to \infty$ and the BL thickness tends to infinity as $\sqrt{a(\zeta)}$: the logarithmic singularity type takes place at $k = 1$ and it is of the power type at $k > 1$. At $k \geq 1$ the flow separation is observed in experimental and numerical studies, this phenomenon leads to change not only the outer part but also the inner boundary-layer structure. It should be noted, such behaviour of velocity viscous perturbations near the BL outer part at the separation development is new property in the comparison with the 2D flows.

The function $W(\eta,\zeta)$ has irregular but finite limit in the leeward plane at $\zeta \to 0$ and $k < 1/3$. This limit is singular at $k \geq 1/3$: the singularity has the logarithmic or power type, if $k = 1/3$ or $k > 1/3$. At $1/3 \leq k < 1$ the singularity is related with the behaviour of cross-flow velocity only. This singularity leads to the longitudinal vortex component strengthening in the outer part of the viscous region. The singularity takes place, if the pressure gradient is negative ($k \leq 2/3$) or positive ($k > 2/3$). It is formed by BL proper solutions, which have homogeneous conditions on both boundaries and arise near the runoff plane. The critical value $k_c = 1/3$ for the outer BL part is undependable on the wall temperature, Mach and Prandtl numbers, however the considered singularities define the real flow structure near the leeward plane at $k \geq 1/3$ [18, 19].

Due to the irregularity of solutions already at $k \geq 1/6$ ($m \leq 2$)the vortex boundary region near the runoff plane is formed with transverse dimension $\zeta \sim \varepsilon^{\frac{1}{2-m}}$; at $m \sim 1$ this value is of the order of the BL thickness ε. In this region, the transverse diffusion is the effect of the first order, and to describe it we introduce variables:

DOI: 10.13009/EUCASS2017-82

$$z = \sqrt{kx}R\zeta/\varepsilon_1, \quad u = u(y,z), \quad h = h(y,z), \quad w = w(y,z), \quad \varepsilon_1 = \left[\tfrac{3}{2}\mathrm{Re}\,\rho_e(\varphi_1)u_e(\varphi_1)/\mu_e(\varphi_1)\right]^{-\frac{1}{2}}.$$

Using these variables from Navier-Stokes equations at $\zeta \sim \varepsilon_1 << 1$ for this region we derive self-similar equations, which in its outer part, at $y >> 1$, reduce to the form:

$$\begin{gathered} U_{yy} + kU_{zz} + (1-k)yU_y + kzU_z = 0, \\ W_{yy} + kW_{zz} + (1-k)yW_y + \left(\frac{2}{z} + kz\right)W_z + 2k(m-1)W + \tfrac{2}{3}p_1U = 0. \end{gathered} \tag{2.6}$$

For $k < 1$ these equations have the solution corresponding to the regular at $k \to 0$ solution of BL equations:

$$\begin{gathered} U(y,z) = C_1 erfc\left(y\sqrt{(1-k)/2}\right) erf\left(z/\sqrt{2}\right), \quad W = -B(z)C_1 erfc\left(y\sqrt{(1-k)/2}\right), \\ B_{zz} + \left(\frac{2}{z} + z\right)B_z - 2(m-1)B = -2mp_1F(z), \quad F(z) = erf\left(z/\sqrt{2}\right). \end{gathered} \tag{2.7}$$

The function $B(z)$ is expressed by Kummer function $\Phi(a,b,x)$ [22]:

$$B = mp_1B_0(z) + B_m\Phi\left(1-m, \tfrac{3}{2}, -\tfrac{1}{2}z^2\right), \quad B_m = b_m\left(R\sqrt{kx}/\varepsilon_1\right)^{2(1-m)}. \tag{2.8}$$

Here $B_0(z)$ is particular solution of the inhomogeneous equation (2.7); the coefficient B_m is determined from matching condition of (2.8) with (2.5). In Fig. 2, comparisons of solutions of BL (dotted lines) and Navier-Stokes (solid lines) equations for $m = 1/2$ (curves 1 and 2) and $m = 1$ (curves 3 and 4) are presented. It is seen, regular solutions of Navier-Stokes equations are converged quickly to singular solutions of BL equations.

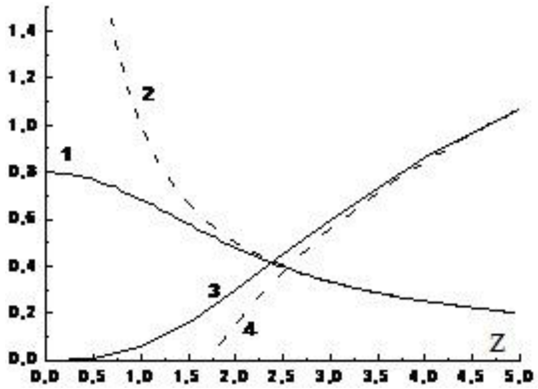


Figure 2: Solutions of boundary layer equations (dotted lines) and parabolized Navier-Stokes equations (solid lines)

Another effect generated by the singularity at $k \geq 1/3$ due to the BL growth at $\zeta \to 0$ is the viscous-inviscid interaction. This effect is important in the region, where the inviscid and induced cross-velocities have same orders; this condition defines the transverse dimension of the region $\Delta\varphi$ and the velocity scale inside it as:

$$\Delta\varphi \sim \sqrt{\varepsilon}x^{-\frac{1}{4}}, \; w_e \sim kRu_e\sqrt{\varepsilon}x^{-\frac{1}{4}}.$$

In this region, the flow has the two-layer structure. Assuming the potential flow in the outer inviscid region the solution here is presented by the improper integral from the displacement thickness $\delta(x,s)$. In the boundary layer the flow is described by full 3D equations:

DOI: 10.13009/EUCASS2017-82

$$s=\frac{R\zeta}{\sqrt{\varepsilon}},\quad w_e=\tfrac{3}{2}u_e\sqrt{\varepsilon}W_e\left(x,s\right),\quad W_e\left(x,s\right)=-ks\left[1+r\right],\quad r=\frac{4m}{\pi}\frac{\partial}{\partial x}\int_0^\infty\frac{\delta\left(x,t\right)dt}{s^2-t^2},$$
$$v=f+Kg+Ag_s+\tfrac{2}{3}xf_x,\quad u_{yy}=W_e wu_s+vu_y+\tfrac{2}{3}xuu_x,\quad h=h_w+h_r u-\tfrac{1}{2}M_e\left(\varphi_1\right)u^2, \tag{2.9}$$
$$w_{yy}=W_e ww_s+vw_y+w\left(\tfrac{2}{3}u+W_{es}w\right)-h\left(\tfrac{2}{3}+W_{es}\right)+\tfrac{2}{3}xuw_x.$$

For these equations boundary conditions have the form (1.1). A solution of Eqs. (2.9) will be matched with (2.3) at $s\to\infty$. For Eqs. (2.9) initial conditions are needed at some streamwise location $x=x_0$, which can be obtained from a solution of Navier-Stokes equations near the body nose; this feature does the problem more complicated. Eqs. (2.9) allow a self-similar solution for hypersonic flows at some additional assumptions.

The solution of Eqs. (2.9) in the outer boundary layer part, at $y>>1$, is described by formulas:

$$t=y\big/\sqrt{d\left(x,s\right)},\ u=1+U\left(x,t,s\right),\ w=1-c\left(x,s\right)U,\ v=y\left[1-k\left(1+r\right)\right],\ U=C_1 erfc\left(t\big/\sqrt{2}\right),$$
$$\left(1+r\right)sd_s-2mxd_x-2\left(n-1-r_s\right)d=-2n,$$
$$\left(1+r\right)sc_s-2mxc_x-2\left(m-1-r_s\right)c=-2m\left(p_1-qp_0\right),\quad p_0=\tfrac{3}{2}\left(\tfrac{1}{2}M_0+h_w-1\right).$$

Along characteristics $\xi\left(x,s\right)=const$, which are streamlines of the inviscid flow, the equations for functions $d=d\left(\xi,s\right)$ and $c=c\left(\xi,s\right)$ are reduced to ordinary differential equations and are integrated. At $s\to 0$ these functions are represented in the form:

$$c=Cs^L+\frac{m\left(p_1+p_0 r_s\right)}{m-1-r},\quad L\left(\xi,s\right)=\frac{m-1-r_s}{1+r};\quad d=Ds^I+\frac{n}{n-1-r_s},\quad I\left(\xi,s\right)=\frac{n-1-r_s}{1+r}, \tag{2.10}$$
$$C=b_m\varepsilon^{m-1},\quad D=a_n\varepsilon^{n-1}.$$

Coefficients C and D obtained by matching for $d\left(\xi,s\right)$ and $c\left(\xi,s\right)$ at $s\to\infty$ with relations (2.5) for $a\left(\zeta\right)$ and $b\left(\zeta\right)$ at $\zeta\to 0$ [18, 19]. The logarithmic singularity appears in these functions at $I=0$ or $L=0$. At $L\left(\xi,0\right)<0$ or $I\left(\xi,0\right)<0$ the singularity is of the power type. It is followed from presented results, the viscous-inviscid interaction doesn't eliminate the singularity, this effect moves only the critical value of k_c.

4. Singularities in boundary layer near-wall region

In the outer BL part, the critical value $k_c=1/3$, although calculations show $k_c=k_c\left(\mathrm{M}_\infty,\mathrm{Pr},h_w\right)$. This indicates that the singularity can arise firstly in the near wall region. To study such possibility, at the first, we study the behaviour of the Eqs. (1.2) solution at $y<<1$ in the runoff plane $\varphi=\varphi_1$. In the near-wall region, the solution can be presented in the following form [21]:

$$f_0=\tau_0\frac{y^2}{2}+F_0(y),\quad g_0=\theta_0\frac{y^2}{2}+G_0(y),\quad \tau_0=\frac{du_0\left(0\right)}{dy},\quad \theta_0=\frac{dw_0\left(0\right)}{dy},$$
$$u_0=\tau_0 y+U_0(y),\quad w_0=\theta y+W_0(y),\quad v_0=-\alpha y^2-F_0+kG_0,\ \alpha=\tfrac{1}{2}\left(\tau_0-k\theta_0\right), \tag{3.1}$$
$$h_0=h_w+h_r\tau_0 y-M_e\tau_0^2y^2+\left(h_r-2M_e\tau_0 y\right)U_0-M_eU_0^2.$$

Using Eqs. (1.2) second terms of these decompositions can be presented by series:

$$U_0=F_{0y}=\sum_{i=0}\frac{\alpha_i y^{i+4}}{(i+4)!},\quad F_0=\sum_{i=0}\frac{\alpha_i y^{i+5}}{(i+5)!},\quad W_0=G_{0y}=\sum_{i=0}\frac{\beta_i y^{i+2}}{(i+2)!},\quad G_0=\sum_{i=0}\frac{\beta_i y^{i+3}}{(i+3)!}. \tag{3.2}$$

DOI: 10.13009/EUCASS2017-82

First three coefficients of these series are defined by relations:

$$\alpha_0 = -2\tau_0\alpha,\ \alpha_1 = k\tau_0\beta_0,\ \alpha_2 = k\tau_0\beta_1,$$
$$\beta_0 = -ph_w,\ \beta_1 = -p\tau_0 h_r,\ \beta_2 = \tfrac{1}{3}\left(\tau_0 - 3k\theta_0\right)\theta_0 + 2pM_e\tau_0^2. \qquad (3.3)$$

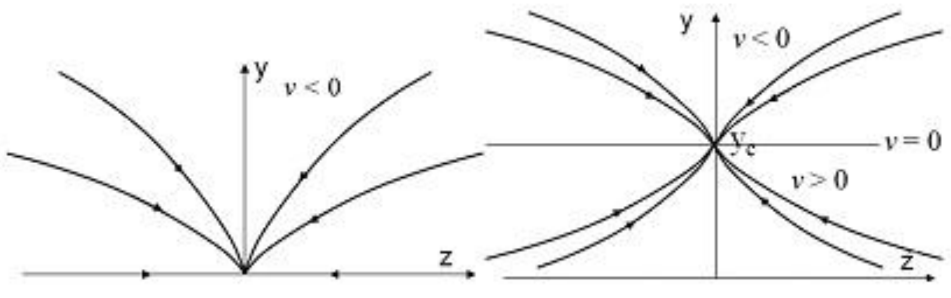
Using these expansions we can study qualitatively a dependence from parameters of the flow structure near the runoff plane by analyzing the behaviour of subcharacteristics of Eqs. (1.2). The transformed normal to the body surface v and transverse w velocities at $\zeta << 1$ and $y << 1$ in the first order approximation are represented in the form:

$$v = v_0 = -\left(\alpha y^2 - \frac{1}{6}k\beta_0 y^3\right) = -\frac{1}{6}k\beta_0 y^2\left(y + y_c\right),\quad y_c = -\frac{6\alpha}{k\beta_0} = \frac{6\alpha}{kph_w},\quad w = -w_0 = -k\theta_0\zeta y.$$

In the plane $\zeta = 0$ the cross-flow velocity $w = 0$ due to symmetry conditions, and two critical points can be here, in which $v = 0$. The first point locates on the cone surface $y = 0$, and the second one $y = -y_c$ appears in the physical space at $\alpha < 0$, if $p > 0$ ($k < 2/3$), that corresponds to small angles of attack for the round cone, and at $\alpha > 0$, if $p < 0$. Commonly, the critical value of the cross-flow velocity gradient $k_c \le 1/3$ and corresponds to the negative cross-flow pressure gradient, when $p > 0$; the transverse skin friction in this region $\theta_0 > 0$. Using these expressions, the equation for subcharacteristics is obtained in the form:

$$\frac{y_c dy}{y\left(y + y_c\right)} = \beta\frac{d\zeta}{\zeta},\quad \beta = \frac{\alpha}{k\theta_0};\quad \alpha \ne 0: y = \frac{y_c y_0 s^{\beta}}{y_c + y_0\left(1 - s^{\beta}\right)},\quad s = \left|\frac{\zeta}{\zeta_0}\right|,$$
$$\alpha = 0: y = \frac{y_0}{1 - y_0 d\ln s},\quad d = \frac{ph_w}{6\theta_0}.$$

Here y_0 and z_0 define the initial point in the cross plane.



a) $\alpha \ge 0$; б) $\alpha < 0$;

Figure 3: Subcharactristics in the cross-plane at $\alpha \ge 0$ (a) and $\alpha < 0$ (b); $p > 0$

The subcharacteristic behaviour is shown in figures 3a and 3b for $p > 0$. At $\alpha > 0$ velocities $v < 0$ and $w < 0$; the only critical point-node is in the coordinate origin, and subcharacteristics go to it from the region $\zeta \ne 0$ (see Fig. 3a). At $\alpha = 0$ $y_c = 0$ and the point $\zeta = y = 0$ is double critical point of the type the saddle-node; the saddle is in the lower half-plane, i.e. out of the physical space. The node is in the upper half-plane and the subcharacteristics pattern retains the same form as at $\alpha > 0$. At $\alpha < 0$ the node drifts in the point $\zeta = 0$, $y = -y_c > 0$, and the coordinate origin becomes by the saddle point (see Fig. 3b). In this case, at $y > -y_c$ the normal velocity $v < 0$ and at $0 < y < -y_c$, $v > 0$; $v = 0$ on the plane $y = y_c$.

DOI: 10.13009/EUCASS2017-82

This analysis shows the physical flow structure varies qualitatively, if the sign of the parameter α changes. The value $\alpha = 0$ is the criterion of the new flow properties appearance. It should be noted that in solutions of Navier-Stokes equations for similar problems near the coordinate origin $\zeta = y = 0$ the streamwise oriented vortex arises, and the flow isn't described by the BL theory. On the base of this qualitative analysis it can suppose that the critical value $k_c\left(h_w, \mathrm{M}\right)$ is defined by the relation:

$$2\alpha\left(k_c\right) = \tau_0\left(k_c\right) - k_c\theta_0\left(k_c\right) = 0\,. \tag{3.4}$$

To support this hypothesis we analyze equations for functions $U_0(y)$ and $W_0(y)$, which are derived by substituting decompositions (3.1) to Eqs. (1.2). Considering functions $U_0(y)$ and $W_0(y)$ as perturbations we can linearize equations and obtain in the first order approximation:

$$\begin{gathered} U_{0yy} + \alpha y^2 U_{0y} + \tau_0\left(F_0 - kG_0\right) = -\alpha\tau_0 y^2, \\ W_{0yy} + \alpha y^2 W_{0y} - \tfrac{2}{3}\left(\tau_0 - 3\theta_0\right) y W_0 + \theta_0\left(F_0 - kG_0\right) = \beta_0 + \beta_1 y + \frac{1}{2}\beta_2 y^2 + \left[\tfrac{2}{3}\theta_0 y - p\left(h_r - 2M_e\tau_0 y\right)\right]U_0. \end{gathered} \tag{3.5}$$

At $y \to 0$ $U_0(y)$ and $W_0(y)$ are expressed by series (3.3); in order to match these functions with the solution of equations (1.2) in the main BL part they will growth at $y \to \infty$ not faster than a power function. Equations (3.5) have the large order and it is difficult to find their analytical solution. To study their solution behaviour at $y \to \infty$ and $\alpha \neq 0$, we introduce the new variable:

$$\xi = -\alpha y^3/3, \quad y = -\left(3\xi/\alpha\right)^{\frac{1}{3}}. \tag{3.6}$$

At the limit $\xi \to \infty$, Eqs. (3.5) are reduced in the first order approximation to the form:

$$\begin{gathered} \xi\frac{\partial^2 U_0}{\partial\xi^2} + \left(\frac{2}{3} - \xi\right)\frac{\partial U_0}{\partial\xi} = -\frac{\tau_0}{3}\left(\frac{3\xi}{\alpha}\right)^{\frac{1}{3}}, \quad c = 2\frac{\tau_0 - 3k\theta_0}{9\alpha}, \\ \xi\frac{\partial^2 W_0}{\partial\xi^2} + \left(\frac{2}{3} - \xi\right)\frac{\partial W_0}{\partial\xi} + cW_0 = -\frac{\beta_1}{3\alpha} + \frac{\beta_2}{6\alpha}\left(\frac{3\xi}{\alpha}\right)^{\frac{1}{3}} - \frac{2}{9\alpha}\left(\theta_0 + 3M_e\tau_0\right)U_0. \end{gathered}$$

Solutions of these equations can be represented as:

$$\begin{gathered} U_0 = A_{00}\int_0^y e^{-\frac{1}{3}\alpha s^3} ds + \tau_0\left(\frac{3\xi}{\alpha}\right)^{\frac{1}{3}}, \\ W_0 = B_{00}\Phi\left(-c, \tfrac{2}{3}, \xi\right) + B_{01}\xi^{1/3}\Phi\left(\tfrac{1}{3} - c, \tfrac{4}{3}, \xi\right) - \frac{3\beta_1}{2\left(\tau_0 - 3k\theta_0\right)} - \frac{3\beta_2}{\tau_0 - 9k\theta_0}\left(\frac{3\xi}{\alpha}\right)^{\frac{1}{3}} - \frac{\theta_0 + 3M_e\tau_0}{\tau_0 - 3k\theta_0}U_{00}. \end{gathered} \tag{3.7}$$

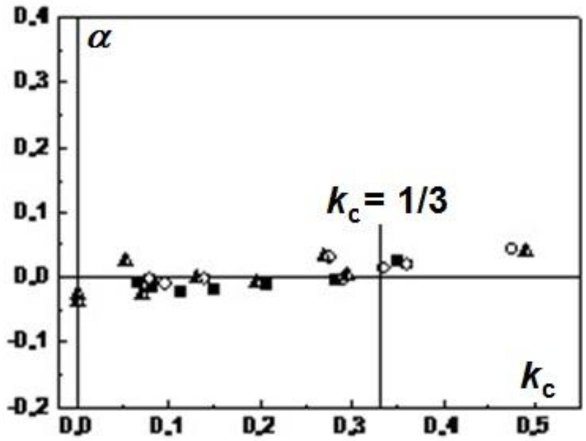
First terms of these expressions are solutions of homogeneous equations, with zero right hand sides; A_{00}, B_{00} and B_{01} are constants; $\Phi(a,b,x)$ is Kummer degenerate hypergeometric function, which has following asymptotes at $\xi \to \infty$ [21]:

$$\alpha > 0,\ \xi < 0 : \Phi \sim \left(-\xi\right)^c; \quad \alpha < 0,\ \xi > 0 : \Phi \sim e^{\xi}\xi^{c-2/3}.$$

Solutions (3.7) growth exponentially at $\alpha < 0$ and $p > 0$; their can't be matched with the solution in the main BL part. Therefore, at these conditions a solution of BL equations can't exist. This conclusion and also the criterion (3.4) for the boundary of the leeward symmetry plane solution existing are confirmed by numerical calculations for the slender round cone at an angle of attack [6 – 12, 20], a part of which is presented in Fig. 4. In this figure, symbols correspond to calculations of limit values $\alpha\left(k_c\right)$ for the solution existing at different boundary conditions in the diapason of Mack numbers from 2 to ∞ at $\mathrm{Pr} = 1$ and for different surface temperatures. At $k < 1/3$ data are

DOI: 10.13009/EUCASS2017-82

grouped near the value $\alpha = 0$ in accordance with Eq. (3.4); apparently, the data scatter is due to the decrease of the calculation accuracy at the approach to the critical value k_c and also with errors of data copying from papers. At $k > 1/3$, all calculations are finished with $\alpha > 0$, since the solution existing in this region is determined by singularities in the outer BL part, but not in the near-wall region.



Figer 4: The boundary of the solution existing in the leeward symmetry plane of the slender co round cone at the angle of attack and $\Pr = 1$ in the dependence of the critical value k_c : ▲ – [11], ■ – [12], ○ – [20].

Then we consider the solution behaviour of full BL equations in the near-wall region beside the runoff plane, at $\zeta << 1$. 3DBL equations have the parabolic type and their solution before the runoff plane know nothing about the solution in this plane, however, in order to the first solution moves smoothly into the last one at $\alpha > 0$, the first will be locally self-similar. Due to this condition, the streamwise $\tau(\zeta)$ and cross-flow $\theta(\zeta)$ friction stresses and the self-similar variable η at $\zeta << 1$ will be defined by expressions:

$$\tau(\zeta) = \frac{\tau_0}{a(\zeta)}, \quad \theta(z) = \frac{\theta_0}{a(\zeta)}, \quad \eta = \frac{y}{a(\zeta)}. \tag{3.8}$$

The function $a(\zeta)$ at $\alpha \geq 0$ will satisfy to the condition $a(0) = 1$. In this case, flow functions in the boundary layer near the wall can be represented in the form:

$$\begin{gathered} f(\eta,\zeta) = a(\zeta)\left[\tau_0 \frac{\eta^2}{2} + F(\eta,\zeta)\right], \quad u(\eta,\zeta) = f_\eta = \tau_0\eta + U(\eta,\zeta), \\ g(\eta,\zeta) = a(\zeta)\left[\theta_0 \frac{\eta^2}{2} + G(\eta,\zeta)\right], \quad w(\eta,\zeta) = g_\eta = \theta_0\eta + W(\eta,\zeta), \\ v = a\left[\left(\alpha - \frac{1}{2}\theta_0 k\zeta \frac{a_\zeta}{a}\right)\eta^2 + F - kG\left(1 + k\zeta\frac{a_\zeta}{a}\right) - k\zeta G_\zeta - k\zeta\eta_\zeta W\right]. \end{gathered} \tag{3.9}$$

Substituting Eqs. (3.8) and (3.9) to Eqs. (1.2) and linearizing the result with respect to disturbances we obtain equations of the first order approximation for the flow in the near-wall region beside the runoff plane:

$$\begin{gathered} U_{\eta\eta} + \alpha\eta^2 U_\eta + a^2\left\{k\theta_0\zeta\eta U_\zeta + \tau_0\left[F - kG\left(1 + \frac{a_\zeta}{a}\right) - k\zeta G_\zeta\right]\right\} = -\alpha\tau_0\eta^2, \\ W_{\eta\eta} + \alpha\eta^2 W_\eta + a^2\left\{k\theta_0\zeta\eta W_\zeta + \theta_0\left[F - kG\left(1 + \frac{\zeta a_\zeta}{a}\right) - k\zeta G_\zeta\right] - 3\alpha c\eta W\right\} = \\ = -\alpha\theta_0\eta^2 + a^2\left\{\beta_0 + \beta_1\eta + \tfrac{1}{2}\beta_3\eta^2 + \left[\tfrac{2}{3}(\theta_0 + 3pM_e\tau_0)\eta - ph_r\right]U\right\} \end{gathered} \tag{3.10}$$

DOI: 10.13009/EUCASS2017-82

Here $\beta_3 = \frac{2}{3}\tau_0\theta_0 - k\theta_0^2 + pM_e\tau_0^2$. In Eqs. (3.10), due to them local self-similarity at $\alpha \geq 0$ at least two first terms will be invariant with respect to transformations (3.8) – (3.9) and will have same coefficients as Eqs. (3.5). This condition defines the function $a(\zeta)$:

$$\alpha a^2 - \frac{1}{2}k\theta_0\zeta a a_\zeta = \alpha, \quad a^2 = 1 + C_2\zeta^q, \quad q = \frac{4\alpha}{k\theta_0}. \tag{3.11}$$

The constant C_2 can be found from comparison with numerical calculations. It is follows from this relation at $\alpha \geq 0$ and $q < 2$ the solution of Eqs. (1.2) in the near wall region at $\zeta << 1$ can find in the form of the decomposition on powers of ζ. First two terms are:

$$\begin{aligned} F(\eta,\zeta) &= F_0(\eta) + \zeta^q F_q(\eta) + ..., \quad U(\eta,\zeta) = U_0(\eta) + \zeta^q U_q(\eta) + ..., \\ G(\eta,\zeta) &= G_0(\eta) + \zeta^q G_q(\eta) + ..., \quad W(\eta,\zeta) = W_0(\eta) + \zeta^q W_q(\eta) + ..., \end{aligned} \tag{3.12}$$

The first term of this expansion coincides by the form with the solution (3.7) for the runoff plane, but depends on the self-similar variable (3.8). Second terms define the proper solution of BL equations (1.2) при $\zeta << 1$, which is found as a solution of the following equations:

$$\begin{gathered} U_{q\eta\eta} + \alpha\eta^2 U_{q\eta} + 4\alpha\eta U_q + \tau_0\left[F_q + F_0 - k(q+1)(G_0 + G_q)\right] = \\ = -\left[F_q + F_0 - k(q+1)(G_0 + G_q)\right]U_{0\eta} - (F_0 - kG_0)U_{q\eta} - kqW_0U_q, \\ W_{q\eta\eta} + \alpha\eta^2 W_{q\eta} - 3\alpha\left(c - \frac{4}{3}\right)\eta W_q - 3\alpha c\eta W_q = \\ = -\theta_0\left[F_q + F_0 - k(q+1)(G_0 + G_q)\right] + \beta_0 + \beta_1\eta + \tfrac{1}{2}\beta_3\eta^2 + \\ + \left[\tfrac{2}{3}(\theta_0 + 3pM_e\tau_0)\eta - ph_r\right](U_0 + U_q) - \\ - \left[F_q + F_0 - k(q+1)(G_0 + G_q)\right]W_{0\eta} - (F_0 - kG_0)W_{q\eta} - kqW_0W_q. \end{gathered}$$

To extract asymptotic forms of functions $U_q(\eta)$ and $W_q(\eta)$ at $\eta \to \infty$ we use the transformation similar to (3.6):

$$\xi = -\alpha\eta^3/3, \quad \eta = -(3\xi/\alpha)^{\frac{1}{3}}.$$

Using this transformation at $\eta \to \infty$ in the first order approximation we obtain equations:

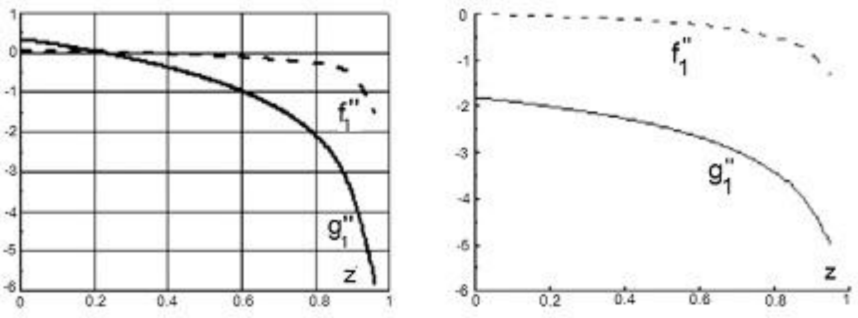
$$\begin{gathered} \xi U_{q\xi\xi} + \left(\frac{2}{3} - \xi\right)U_{q\xi} - \frac{4}{3}U_q = 0, \quad c = 2\frac{\tau_0 - 3k\theta_0}{9\alpha}, \\ \xi W_{q\xi\xi} + \left(\frac{2}{3} - \xi\right)W_{q\xi} + \left(c - \frac{4}{3}\right)W_q = -\frac{\beta_1}{3\alpha} + \frac{\beta_3}{6\alpha}\left(\frac{3\xi}{\alpha}\right)^{\frac{1}{3}} - \frac{2(\theta_0 + 3pM_e\tau_0)}{9\alpha}(U_0 + U_q). \end{gathered}$$

The solution of these equations is expressed by formulas:

$$\begin{gathered} U_q(\xi) = A_{q0}\Phi\left(\frac{4}{3}, \frac{2}{3}, \xi\right) + A_{q1}\xi^{\frac{1}{3}}\Phi\left(\frac{5}{3}, \frac{4}{3}, \xi\right), \\ W_q(\xi) = B_{q0}\Phi\left(\tfrac{4}{3} - c, \tfrac{2}{3}, \xi\right) + B_{q1}\xi^{1/3}\Phi\left(\tfrac{5}{3} - c, \tfrac{4}{3}, \xi\right) + \frac{9\beta_1}{2\tau_0} + \frac{3\beta_2}{\frac{11}{3}\tau_0 - k\theta_0}\left(\frac{3\xi}{\alpha}\right)^{\frac{1}{3}} + \frac{\theta_0 + 3M_e\tau_0}{2\tau_0}U_0 - \frac{\theta_0 + 3M_e\tau_0}{\tau_0 - 3k\theta_0}U_q. \end{gathered}$$

Here A_{q0}, A_{q1}, B_{q0} and B_{q1} are constants. These relations show that the proper solution in near-wall BL region near the runoff plane is nonzero. It is irregular at $\alpha \geq 0$ and it is singular at $\alpha < 0$. The logarithmic singularity is not in this case, and the solution of BL equations exists at the critical value k_c in contrast to the outer region.

DOI: 10.13009/EUCASS2017-82



a) b)

Figer 5: Skin friction distributions on the small aspect ratio delta wing at $M_\infty = 2$ related with:
a) the second boundary-layer approximation; b) the angle of attack

Numerically singularities in solutions of 3DBL equations were found in the turbulent flow on the swept wing in the form of finite velocity jump [15] and at the asymptotic laminar flow analysis around slender delta wings with the small aspect ratio in the supersonic freestream [20, 21]. In the work [21], at the analysis of perturbations related with the angle of attack and the boundary layer it was found that they lead to infinite disturbances in the symmetry plane, although equations no visible singularities contained. In this case, the first order approximation is described by the Blasius solution for the delta flat plate. In Fig 5, dimensionless longitudinal and transverse skin friction distributions $f_1''(z)$ и $g_1''(z)$, induced by the second order BL approximation (Fig. 4a) and the angle of attack (Fig. 4b) in dependence on transverse coordinate $z = 1 - Z/X$, where X and Z are Cartesian streamwise and span coordinates. Skin friction perturbations infinitely growth, when the symmetry plane ($z = 1$) is approached. Detailed investigation of equations for these functions shows that in these cases singularities take place as in the near-wall and outer BL parts. In the outer part, the singularity corresponds to values of the parameter $m = 3/4$ and 7/8 in relations (2.5) for cases *a* and *b*, correspondently. The longitudinal velocity perturbation singularity is related only with the near-wall singularity described by decomposition (3.12).

5. Conclusions

In this work, the short review of investigations of singularities in solutions of BL equations, which are formed when two streamline families are collided, is presented. This phenomenon can arise only in unsteady and 3D problems and has not an analogue in 2D flows. A typical example of such problem is the flow around a slender cone in the vicinity of the runoff plane. In this case, solutions are found in the analytical form that allows analyzing explicitly the singularity character.

The analysis of solutions for the outer flow part revealed two singularity types. One type is in streamwise and cross velocity viscous perturbations; it arises at values of relative cross pressure gradient $k \geq 1$ and leads to the exponential disturbance growth, if the runoff plane is approached. At $k = 1$ the singularity is logarithmic and at $k > 1$ it is power type; its appearance is correlated with the BL separation appearance. Another singularity type at smaller values of $k \geq 1/3$ leads to the infinite growth of transverse velocity perturbations only and isn't related directly with the flow separation; at $k = 1/3$ the singularity is logarithmic and at $k > 1/3$ it is power type. These singularities correspond to some asymptotic flow structure at $\mathrm{Re} >> 1$. This structure includes the boundary region with the dimension of the order of the BL thickness, in which the viscous transverse diffusion effect smoothes the singularity. The comparison of obtained solutions of parabolized Navier –Stokes equations describing the flow in the boundary region with solutions of BL equations confirms this conclusion. Second region induced by the viscous-inviscid interaction effect has the transverse dimension of the order of square root from the BL thickness and the two-layer structure. For the potential flow in the outer inviscid subregion the integral solution representation is found on the base of the slender wing theory. The inner subregion is described by full 3DBL equations, the solution of which is obtained for the outer viscous subregion part. It was shown that the viscous-inviscid interaction don't eliminate the singularity but drifts it in the parametric space; to eliminate the irregularity the boundary region is needed.

DOI: 10.13009/EUCASS2017-82

To find the dependence of the critical parameter of the singularity appearance k_c on Mach and Prandtl numbers and the wall temperature solutions of BL equations are studied in the near-wall region beside the runoff plane. Equation subcharacteristics (streamlines) analysis showed the presence of one parameter α, the sign of which defines the qualitative change of the streamline topology and, consequently, the physical flow structure. It is shown and is confirmed by comparison with all available calculations, the boundary of the solution existing in the runoff plane corresponds to the criterion $\alpha(k_c)=0$. The analysis of solutions of BL equations near the runoff plane revealed the presence at $\alpha \geq 0$ of irregular and at $\alpha < 0$ singular proper solutions. This is confirmed by numerical calculations of the flow around slender delta wing with the small aspect ratio. Singularities in the near-wall region generate the some flow structure in its vicinity, the study of which is out of this paper framework. Presented results don't depend on outer boundary conditions and are true for the full freestream velocity diapason including hypersonic flows.

Presented results allow concluding that the flow near symmetry planes, for example, on wings, has the complex structure, which is needed to take into account at the numerical modeling in order to eliminate the accuracy loss. Regular flow function expansions commonly used at solutions of BL equations are not applied near this plane, and it can't consider as a boundary condition plane due to a possible solution disappearance.

Acknowledgments

The work is performed at the support by the RFBR grant N 15-01-03615.

References

[1] Stewartson K. 1951. On the impulsive motion of a flat plate in a viscous fluid. *Q. J. Mech. Appl. Math*. 4: 182–198.
[2] Stewartson K. 1964. The theory of laminar boundary layers in compressible fluids. Oxford, Clarendon press.
[3] Stewartson K., Simpson C.J. 1982. On a singularity initiating a boundary layer collision. *Q. J. Mech. Apppl. Math*. 35: 1–16.
[4] Timoshin S.N. 1991. A singularity in the solution of boundary layer equation at the collision of two near-wall jets. *Izvetstia AS USSR. MZhG*. 4: 75–81.
[5] Williams J.C. 1985. Singularities in solution of three-dimensional boundary layer equations. *J. Fluid Mech.* 160: 257–279.
[6] Moore F.K. 1956. Three-dimensional boundary layer theory. *Advances in Applied Mechanics*. 4: 159–228.
[7] Bashkin V.A. 1968. On uniqueness of self-similar solutions of three-dimensional laminar boundary layer equations. *Izvetstia AS USSR. MZhG*. 5: 35-41.
[8] Bashkin V.A., Dikansky E. A. 2001. Laminar boundary layer on source and sink lines of conical bodies in supersonic freestream at different Prandtl numbers. *Uchen. Zap. TsAGI*. 32, N 3-4: 66-80.
[9] Dwyer H.A. 1971. Boundary layer on a hypersonic sharp cone at small angle of attack. *AIAA J*. 9, N 2: 277–284.
[10] Roux B. 1972. Supersonic laminar boundary layer near the plane of symmetry of a cone at incidence. *J. Fluid Mech*. 51, Pt 1: 1–14.
[11] Wu P., Libby P.A. 1973. Laminar boundary layer on a cone near a plane of symmetry. *AIAA J*. 11, N 3: 326–333.
[12] Murdock J.W. 1972. The solution of sharp cone boundary layer equations in the plane of symmetry. *J. Fluid Mech*. 54, Pt 4: 665–678.
[13] Boericke R.R. 1971. The laminar boundary layer on a cone at incidence in supersonic flows. *AIAA J*. 9. N 3: 462–468.
[14] Neyland V.Ya., Sokolov L.A. 1977. Laminar boundary layer on a cone at angle of attack in supersonic freestream. *Tr. TsAGI*. 1819: 3-9.
[15] Cousteix J. Houdeville R. 1981. Singularities in three–dimensional turbulent boundary–layer calculations and separation phenomena. *AIAA Pap*. 1201.
[16] Rubin S.G., Lin T.C., Tarulli F. 1977. Symmetry plane viscous layer on a sharp cone. *AIAA J*. 15, N 2: 204–211.
[17] V.I. Shalaev. 1993. Singularities in the boundary layer on a cone at incidence. *Izvestia RAN. MZhG*. 6: 25-33.
[18] V.I. Shalaev. 2007. Singularities of 3D laminar boundary layer equations and flow structure near a sink plane on conical bodies. *Fluid Dynamics*. 42, N 4: 560–570.
[19] V.I. Shalaev. 1992. Boundary layer on thin small aspect ratio wings. *Journal of Applied Mechanics and Technical Physics*. 1: 71–78.

DOI: 10.13009/EUCASS2017-82

[20] V.I. Shalaev. 2010. Application of Analytical Methods in Contemporary Aerodynamics. Part 1. Boundary Layer Theory. Moscow, MIPT: 300 p.

[21] V. I. Shalaev. 2016. Singularities of 3D laminar boundary layer equations and flow structure in their vicinity on conical bodies. *International Conference on the Methods of Aerophysical Research (ICMAR 2016). AIP Conf. Proc*. 1770: 030055-1–030055-14.

[22] 1979. Handbook of mathematical functions. Ed.M. Abramowits, I. Stegun. Moscow, Nauka: 832 p.