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New improvements in the optimization of the launcher ascent trajectory through the HJB approach

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Abstract
This study completes [1] presented at EUCASS 2015. We investigate the resolution of the launcher ascent trajectory problem by the so-called Hamilton-Jacobi- Bellman (HJB) approach. The method gives a global optimum without any initialization procedure; nevertheless, the computational cost increases quickly for high dimension problems. To mitigate this, the payload mass was excluded from the parameters managed by the HJB equations. A new formulation is proposed to consider explicitly this parameter, while limiting the computation time. An illustration is proposed on a heavy class launcher, for a GEO mission. This study has been performed in the frame of the CNES Launchers Research & Technology program.

1. Introduction

A brief recall of the assets motivating the application of the HJB approach for launcher trajectory optimization is proposed, together with an overview of the associated principles and of the technical status at the end of the previous study. Then, a theoretical formulation allowing to re-introduce explicitly the payload mass in the HJB resolution is proposed. An implementation and associated challenges are then presented. In the end, the HJB algorithm is operated to optimize the trajectory of a heavy class launcher, for a typical GEO mission; results are analysed with regard to a solution obtained through a shooting method.

For any acronym in the following text, please refer to §"Acronyms" at the end of the paper. For any mathematical notation in the following text, please refer to §"Nomenclature" at the end of the paper.

2. Motivations and previous results

2.1 Mapping of optimization methods and associated issues

Most of optimization methods rely on necessary condition of optimality. In the field of control optimization, we can quote the Pontryagin Maximum Principle [2], which uses relationship between real and dual states, and Direct Approach [3], which can be declined in various discretization strategies of the control and of the state. Approximate approaches can also be considered to simplify the algorithm implementation by looking for the optimal solution in a restrained control shape family, but leading to a sub optimal solution.

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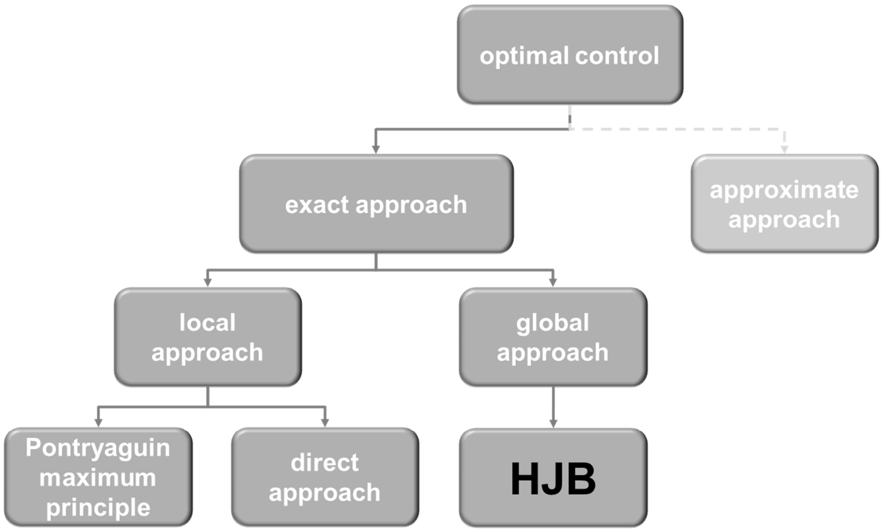

Figure 1: Illustration of classes of optimization approaches

Algorithms derived from exact optimization approaches previously mentioned can be used to find an optimal solution, but it should be emphasized that there is no guaranty that this solution is the global optimal one, because these approaches deal only with necessary conditions of optimality.

Besides, most of these algorithms are typically operated through iterative processes; as a consequence, it raises the question of initialization, and convergence. Finding a good initial guess (in the sense that it would lead to a solution of good quality at the end of the iteration process) can be quite tricky, in particular when dealing with duality and when the area of convergence is restricted. Efforts have also to be put on the iterative process, so as to get an optimal solution in a reasonable time. A detailed presentation of optimization algorithm families can be found in [4].

With regard to local methods, the HJB approach has two major assets: first, the background theory assures to obtain the global solution when it exists. Secondly, the implementation of the HJB approach does not require any iterative process, freeing the engineer from tricky tasks that are initialization and convergence. These great assets justify the motivation for the efforts made jointly by CNES and ENSTA on applying HJB approach to optimize launcher trajectories.

2.2 Elements related to the HJB approach – discrete problems

Let us consider the following problem: we look for the path starting at point a, and allowing reaching point d in minimum time.

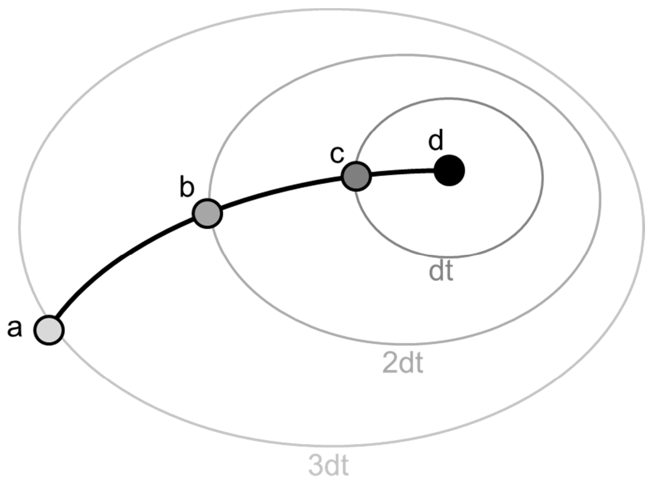

Figure 2: Illustration of the Dynamic Programming Principle

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The dynamic is such that, at any time, the time of arrival depends only on the current state and on the control we choose; in particular, it is considered that the history of dynamic state prior to the current state has no explicit influence on the future state. In this frame, the core idea at the origin of Dynamic Programming [5] may be expressed in this term: if the optimal trajectory from point a to point d goes through point b, the portion of this trajectory linking point b to point d is also the quickest path between these latter points. If it was not, there would exist a quicker trajectory between point a and point d. Now considering the quickest trajectory between point b and point d, this smaller optimization problem can be solved if we know that this trajectory goes through point c, and so on.

This simple-looking idea is at the origin of a powerful approach that can be used to solve an optimal control problem: starting from the target, we look for the points being reachable in a given time dt. The locus of this point may be understood as a level curve, which is memorized. Starting from this curve, the location reachable in a given time dt is determined; this new level curve represents the locus of points reachable in 2dt, and is memorized. This computation is running until the farthest level curve crosses point a. Then, this propagation phase, which is time reverse propagated, stops.

Now, starting from point a, we use the level curves which have been successively memorized during the propagation phase to determine the quickest path, the time flowing in direct sense from 0 to dt, then from dt to 2dt, and so on until reaching point d. This is the reconstruction phase. Let us notice that not only the best path starting from point a can be reconstructed, but all the paths starting from any point included inside the level curve crossing point a.

As previously described, the Dynamic Programming approach proposes to solve an optimal control problem by considering it as a part of a more global family of optimal control problems: to find the optimal path from point a to point d, one may find the optimal path from point b to point d, and so on. The HJB equation extends this approach, expressed in discrete domain, to continuous optimal control problem. As recalled hereafter, solving HJB equation leads to a “value function”, which characterizes the optimal control problem family mentioned before, and which can be determined as the solution of a first order nonlinear Partial Differential Equation which dimension is related to the number of variables involved in the problem.

The resolution of this equation relies on a retro-propagation phase starting from the final target (the injection on the final orbit), and “flowing” back in time on the state domain. Once the value function is computed on the domain, the minimal time trajectory is determined during a reconstruction phase, by “descending” timely (from the launchpad to the final orbit) along the domain path where the value function is the lowest, It is then possible to get the optimal feedback control law and the corresponding optimal trajectory, typically by using a classical second-order Runge-Kutta scheme.

2.4 Challenges related to the implementation of the HJB approach

The HJB equation is set in the whole space $R+\times R_d$, with $d = 6$; nevertheless, in order to perform computations a finite domain is used, on which the HJB equation is discretized. For this a uniform space grid is chosen; the time step is chosen constant for simplicity. Advanced numerical technics are then applied to efficiently evaluate the approximation of the value function, which corresponds to the propagation phase previously described.

Main issues related to HJB approach are due to the strong dependence of computation efforts (in terms of memory size and computation time) to the number of dimensions of the optimization problem to solve: these issues are due to the fact that the HJB equation is a partial differential equation which has to be solved in terms of state and time which all are discretized. Besides, in the frame of the HJB approach, parameters behave like additional states (associated to steady dynamics), which increase the number of dimensions to consider.

When dimension of the problem increases (typically, above dimension 3), numerical computation becomes very challenging. This is what Robert Bellman describes as “the curse of dimensionality”; for a typical three dimensional trajectory such as the one presented here after, 12 dimensions have to be considered (see §”Numerical results”); considering a very rude discretization of 10 points in each dimension (decreasing the computation accuracy), we still get 10exp(12) points to handle. In the frame of launcher trajectory optimizations, efforts have been pushed on dealing with high dimension problems.

During the last study, the analysis of the launcher flight sequential showed that the payload mass was not only the criterion to maximize, but it also intervened as a parameter, which influenced the dynamics of all the flight phases. In order to reduce the dimension of all the sub problems, the HJB resolution was run considering the payload mass as a

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constant. The criterion to be minimized in the HJB problem was the combustion duration, which corresponds to the propellant consumption; this formulation allowed expressing the launcher trajectory optimization problem as a minimum time problem, which matched the notion of reachability inherent to the HJB approach. At the end of the reconstruction, if the optimal trajectory did not use all the available propellant, it meant that the launcher could bring a heavier payload: the above process was run another time with an increased value of the payload mass. The process went on until payload mass variations tended to be negligible. This scheme preserved the properties of HJB approach assuring to reach the global optimal solution. Unfortunately, it also imposed to run iteratively several HJB computations, leading to a significant computational effort.

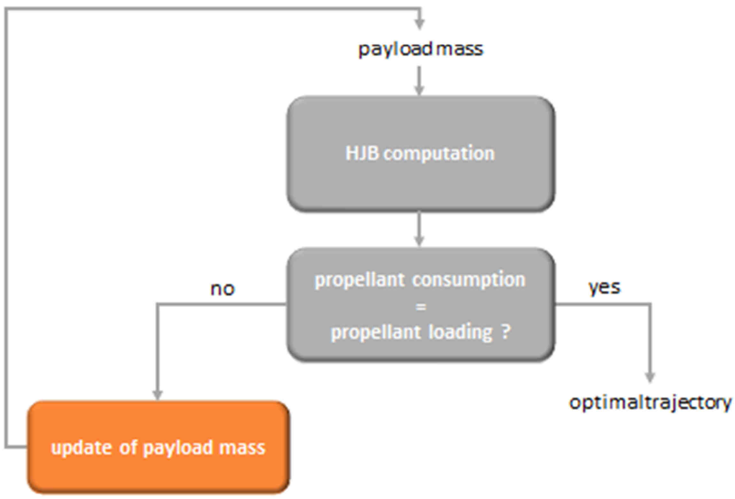

Figure 3: Implementation of the HJB approach

3. Mathematical formulation. Optimal control problem of a hybrid system

A formulation of the trajectory optimization problem suitable for HJB computations implies firstly a precise characterization of the flight sequential, and of the variables to optimize (control, parameters).

3.1 Presentation of the reference problem

A mission toward GEO orbit with an “Ariane5-like” launcher is considered. The choice of this mission is justified by satellite market considerations and also because it includes several elements which are challenging for the HJB approach implementation: the state domain is large (roughly 36 000 km around the Earth, required launcher increase of speed of more than 10 000 m/s), the mission duration is long (more than 5 hours), the number of parameters is also important, as detailed hereafter.

The objective is to optimize the launcher trajectory so as to maximize the payload mass injected on the GEO orbit. The configuration of the launcher relies on two solid propulsion boosters fired on ground burning simultaneously, a cryogenic lower stage ignited on ground and still operating after booster jettisoning, and a cryogenic upper stage. The engine of the upper stage can be ignited several times in flight; the total combustion duration is spread among the different burns. In addition, the perigee altitude of the transfer orbit reached at the end of the first upper stage boost should be at least 180 km. For all the engines, the profile of thrust magnitude is pre-set on ground; the trajectory is commanded through the orientation of thrust.

The launcher is operated from French Guyana; the flight starts with a vertical take-off, goes on with a pitch over maneuver at a constant angle rate which intensity can be optimized, and is followed by a atmospheric gravity turn phase during which the launcher thrust is steered according to a null aerodynamic angle of attack path, so as to limit

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aerodynamic loads on launcher structures. The azimuth of launch, characterizing roughly the geographical direction of the plane in which occurs the atmospheric flight has also to be optimized.

The atmospheric phase (named hereafter "Phase 0") ends with the booster jettisoning; above begins the exo-atmospheric phase (firstly, with the flight of the second stage: "Phase 1") during which the thrust orientation is fully optimized. The upper stage injects the satellite on the GEO orbit thanks to two boosts; respective combustion durations ("Phase 2" and "Phase 4") and associated ballistic phase ("Phase 3") length have to be optimized.

3.2 Introduction of the "consumption time" variable

The final time t_f of the mission is unknown because it depends on the duration of the ballistic flight, τ_B that has to be optimized. As all the propellant of the launcher must be consumed at the end of the mission, the sum of the durations of the phases 1 and 4 of the flight corresponds to the entire consumption of the second E_2 engine, leading to: $(t_2 - t_1) + (t_f - t_3) = M_{P,2}/\beta_{E2}$. Then, the total consumption time is given by:

$$T := \frac{M_{P,E2}}{\beta_{E2}} + t_1 - t_0. \tag{7}$$

The idea now is to introduce a new time variable s that corresponds to the duration of the propellant consumption during the phases 1, 2 and 4 of the flight, as follows:

$$s := \begin{cases} t - t_0 & \text{if } t \in [t_0, t_2[\\ t_2 - t_0 & \text{if } t \in [t_2, t_3[\\ t - t_3 + (t_2 - t_0) & \text{if } t \in]t_3, t_f] \end{cases} \tag{8}$$

and we will denote $s = s(t)$ the above correspondence. It is clear that $s \in [0, T]$ and that $s = 0$ for $t = t_0$. Observe also that $s(.)$ is continuous, and remains constant equal to $s_* = s(t_2) = t_2 - t_0$ for $t \in [t_2, t_3]$. Conversely, we can define $t = t(s)$, then $t(s)$ will have a jump at $s_* = t_2 - t_0$: $t(s_*^-) = t_2$, and $t(s_*^+) = t_3$. In the sequel we will denote $s_* := t_2 - t_1$ the duration of the first boost of the second stage engine. This value is an optimization variable. The definition of the consumption time variable s is illustrated hereafter:

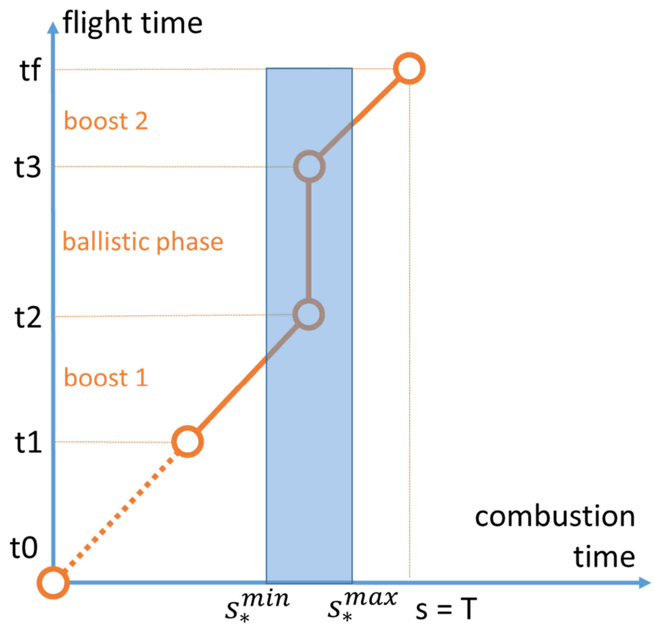

Figure 4: Relations between the physical time variable t and the "consumption" variable s

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3.3 Dynamical system and state transfer function

Let $x := (r, \ell, v, \gamma, \chi) \in R^5$ be the physical state vector. and let $m \in R_+$ be the payload mass. This mass is constant during the flight; hence its evolution equation is simply:

$$\dot{m} = 0 \tag{9}$$

The fundamental principle of dynamics allows us to express:

$$\dot{x} := f(x, u) \tag{10}$$

with u(.) the launcher control

Let $y := (x, m) = (r, \ell, v, \gamma, \chi, m) \in R^6$ be the state vector of the system. First, we introduce the transfer function for the ballistic flight phase. During this phase, the thrust force is zero, because the upper stage engine is off, and the right hand side of the equations (10) does not depend on the total mass of the system {launcher+payload} M anymore. Therefore, the launcher's motion is governed by an uncontrolled and autonomous differential system:

$$\begin{cases} z'(t) = \varphi(z(t)), \quad t \in [t_2, t_3] \\ z(t_2) = y_0 \end{cases} \tag{11}$$

where $\varphi(z)$ denotes the function of the right hand side of (10) with no thrust force

Now, introduce the associated flow map $\Phi(t, y_0)$, so that the solution of (11) satisfies $z(t) = \Phi(t, y_0)$. The application Φ can be considered as a state transfer function that associates to a terminal state of the phase 2 of the flight $z(t_2) = y(s(t_2)^-) = y(s_*^-)$ the starting state of phase 3 $z(t_3) = y(s(t_3)^+) = y(s_*^+)$, by:

$$y(s_*^+) = \Phi(\tau_B, y(s_*^-)) \tag{12}$$

where τ_B is the duration of the ballistic flight

Now, during phases 1, 2 and 4 (i.e., for $t \notin [t_2, t_3]$), the system is controlled by $u = (\alpha, \delta)$, the two orientation angles of the thrust force. By using the time variable transformation (8) and the fact that $M(t) = M(t(s))$, we can write all time-dependent quantities of equations (10) as functions of the variable s. Therefore the system of differential equations (9) and (10) can be re-written as:

$$\begin{cases} \mathbf{y}'(s) = f(s, \mathbf{y}(s), \mathbf{u}(s)), \quad s \in [0, s_*[\\ \mathbf{y}(s_*^+) = \Phi\big(\tau_B, \mathbf{y}(s_*^-)\big), \\ \mathbf{y}'(s) = f(s, \mathbf{y}(s), \mathbf{u}(s)) \quad s \in]s_*, T] \end{cases} \tag{13}$$

where the function $f(.)$ represents the right hand sides of (9) and (10), and s is the consumption time variable.

The combined notions of combustion duration and of flow map leads to (13), which can be interpreted as a hybrid system: the link between the kinematic conditions at the extremities of the ballistic phase (which is only controlled by its duration) can be evaluated separately from the dynamics of the propelled phases occurring before and after this ballistic phase. As soon as the flow map $\Phi(t, y_0)$ is computed, the relationship between the kinematic conditions at the extremities of the ballistic phase is known: the dynamics "switches" instantaneously from the first to the second upper stage boost. As a consequence, the computations presented in §5 can be computed separately for these propelled phases, the global physical consistency and optimality of the trajectory being assured thanks to the formulation proposed in §4, which takes into account the hybrid dynamics proposed here.

3.4 State constraints

Because the target orbit is at a low altitude, a special constraint on the dynamic thermal flow has to be satisfied during the phase 2 of the flight:

$$0.5\,\rho(r)v^3 \leq 555.0\ W/m^2 \tag{14}$$

where $\rho(r)$ is the density of the atmosphere at altitude r

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The model of density for atmosphere at high altitude used in this work is defined by a tabulated function. The value of the constraint is arbitrarily set to a value which order of magnitude may be representative with a standard mission analysis. Then the set of state constraints, in R^6, is defined by:

$$K \quad := \quad \{y = (x, m) \in R^6, \quad 0.5\,\rho(r)v^3 \leq 555\} \tag{15}$$

3.5 Target set

The target set is the GEO orbit. In order to represent it in the space of spherical coordinates $(\mathrm{r}, \mathrm{L}, \ell, \mathrm{v}, \gamma, \chi) \in \mathrm{R}^6$, standard formulas from orbital mechanics are used. In particular one can express the eccentricity $\mathrm{e}(\mathrm{x})$, the major semi-axis $\mathrm{a}(\mathrm{x})$ and the inclination $\mathrm{i}(\mathrm{x})$ as functions of the five spherical coordinates $\mathrm{x} \in \mathrm{R}^5$. Then the target set, in R^6, is defined by:

$$\mathrm{C} := \{y = (x, m) \in R^6,\ \mathrm{s.t.}\ e(x) = 0, a(x) = 42168000.0\ \mathrm{m},\ i(x) = 0.0\ \mathrm{rad},\ m \geq 0\} \tag{16}$$

3.6 Optimal control problem

Denote $\mathrm{y}_{\mathrm{y}}^{\mathrm{u}}(\cdot) = (\mathrm{x}_{\mathrm{y}}^{\mathrm{u}}(\cdot), \mathrm{m}_{\mathrm{y}}^{\mathrm{u}}(\cdot))$ the solution of (13), with initial data $\mathrm{y}_{\mathrm{y}}^{\mathrm{u}}(0) = \mathrm{y}$, and for controls $\mathrm{u}(\cdot)$ in a the set of admissible controls U_{ad}:

$$\begin{gathered}\mathrm{U}_{\mathrm{ad}} := \{\mathrm{u}: (0, \mathrm{T}) \to \mathrm{R}^2\ \text{measurable}, \mathrm{u}(\mathrm{s}) \in \mathrm{U}\ \text{a.e.}\}.\\ \text{with}\ \mathrm{U} := [0, 2\pi] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\end{gathered} \tag{17}$$

The optimal control problem (P) corresponding to the GEO mission can be formulated as follows:

$$\begin{cases}\sup m_y^u(T)\\ \text{s.t.}\\ (i) y_y^u = (x_y^u, m_y^u)\ \text{is solution of (13) associated to}\ u\\ \text{with}\ y(0) = y \in X_0 \times [0, +\infty[,\\ (ii) y_y^u(s) \in K, \quad \forall s \in [0, T],\\ (iii) s_* \in [s_2^{min}, s_2^{max}]\ \text{and}\ \tau_B \in [0, \tau_B^{max}]\\ \ \text{with}\ y(s_*^+) = \Phi(\tau_B, y(s_*^-))\\ (iv) y_y^u(T) \in C.\end{cases} \tag{18}$$

As mentioned earlier, the duration of the ballistic flight τ_{B} may be equal to 0. The control problem can be seen as a control of a hybrid system where the control strategy includes (at most) one possible switch. The switching is controlled by $\tau_{\mathrm{B}} \in [0, \tau_{\mathrm{B}}^{\max}]$, the duration of the ballistic flight. The value of the bounds $\mathrm{s}_2^{\min}, \mathrm{s}_2^{\max}$ and $\tau_{\mathrm{B}}^{\max}$ can be chosen by using some physical considerations. For example, one can consider $\mathrm{s}_2^{\min} = \mathrm{t}_1 - \mathrm{t}_0$, $\mathrm{s}_2^{\max} = \mathrm{T}$ and $\tau_{\mathrm{B}}^{\max} = \mathrm{T}$.

4. HJB approach

The method that will be used to solve the problem (P) is based on the definition of two reachability problems associated with different phases of the flight. In all the sequel, for any subset S of R^6 with a boundary $\partial\mathrm{S}$, d_{S} denotes the signed distance function to S, defined by $\mathrm{d}_{\mathrm{S}}(\mathrm{y}) = \mathrm{dist}(\mathrm{y}, \partial\mathrm{S})$ if $\mathrm{y} \in \mathrm{S}$, and $\mathrm{d}_{\mathrm{S}}(\mathrm{y}) = -\mathrm{dist}(\mathrm{y}, \partial\mathrm{S})$ otherwise.

4.1 A reachability problem associated to the second boost of the second stage

From now on, for any $\mathrm{s} \in [0, \mathrm{T}]$ and $\mathrm{y} \in \mathrm{R}^6$, we denote by $\mathrm{y}_{\mathrm{s,y}}^{\mathrm{u}}(\cdot) = (\mathrm{x}_{\mathrm{s,y}}^{\mathrm{u}}(\cdot), \mathrm{m}_{\mathrm{s,y}}^{\mathrm{u}}(\cdot))$ the trajectory starting in y at the initial time s and associated to $\mathrm{u}(\cdot) \in \mathrm{U}_{\mathrm{ad}}$:

$$\begin{cases}\mathbf{y}'(\xi) = f(\xi, \mathbf{y}(\xi), \mathbf{u}(\xi)), \quad \xi \in [0, s_*[\\ \mathbf{y}(s_*^+) = \Phi(\tau_B, \mathbf{y}(s_*^-))\\ \mathbf{y}'(\xi) = f(\xi, \mathbf{y}(\xi), \mathbf{u}(\xi)), \quad \xi \in]s_*, T]\\ \mathbf{y}(s) = y\end{cases} \tag{18}$$

Let us define the value function, for $s \in [s_*^{min}, T]$ and $y \in R^6$, by:

$$w_0(s,y) = \inf_{u \in U_{ad}} \left\{ d_C\left(y_{s,y}^u(T)\right) \vee \max_{\xi \in [s,T]} d_K\left(y_{s,y}^u(\xi)\right) \right\} \quad (19)$$
$$\text{(where } a \vee b := \max(a,b))$$

The control problem (19) corresponds to a classical maximum running cost problem with fixed horizon T. Problem (19) has no "explicit" state constraint. In fact, in this new setting, the term max $d_K\left(y_{s,y}^u(\xi)\right)$ plays the role of a penalization that a trajectory would pay if it violates the state constraints. The advantage of considering (19) is that the function w_0 can now be characterized as the unique continuous solution of an HJB equation; indeed, from [6] and [7], the following result holds.

Theorem 2 (i) The function w_0 is the unique Lipschitz continuous viscosity solution of the following HJB equation on $[s_*^{min}, T] \times R^6$:

$$\min(-\partial_s w_0(s,y) + H(s,y,D_y w_0(s,y)), w_0(s,y) - d_K(y) = 0,$$
$$w_0(T,y) = d_C(y) \vee d_K(y)$$
$$\text{where for any } s \in [0,T],\ y \in K \text{ and } q \in R^6\text{: } H(s,y,q) := \max_{u \in U} -\langle f(s,y,u), q \rangle \quad (20)$$
(and where $\partial_s w$ and $D_y w$ represent respectively the time and space derivatives, and U is defined as in (17))

Here, d_K is a "fictitious cost" that a trajectory would pay if it leaves K; it comes from the presence of the sup-norm term $\max_{\xi \in [s,T]} d_K\left(y_{s,y}^u(\xi)\right)$in the cost function which defines w_0.

(ii) Moreover, we have:

$$w_0(s,y) \le 0 \quad \Leftrightarrow \quad \forall \varepsilon > 0,\ \exists u_\varepsilon \in U_{ad},\ d_C(y_{s,y}^{u_\varepsilon}(T)) \le \varepsilon$$
$$\text{and } d_K(y_{s,y}^{u_\varepsilon}(\xi)) \le \varepsilon\ \forall \xi \in [s,T] \quad (21)$$

For a detailed definition of the viscosity notion, please refer to the monograph of [8]. Assertion (ii) amounts to saying that if $w_0(s,y) \le 0$ then there exists a trajectory $y_{s,y}$ that is as close as desired to the target C.

4.2 A reachability problem associated to problem $(\boldsymbol{P})$

Now, define a more general control problem, for $s \in [0,T]$ and $y \in R^6$, by:

$$w(s,y) = \inf_{u \in U_{ad}} \left\{ d_C\left(y_{s,y}^u(T)\right) \vee \max_{\xi \in [s,T]} d_K\left(y_{s,y}^u(\xi)\right) \right\} \quad (22)$$

To characterize the value function w as a solution of HJB equation, we need to introduce the "jump" operator M defined, for any Lipschitz continuous function σ, by:

$$M\sigma(s,y) := \min_{\tau \in [0, \tau_B^{max}]} \sigma(s, \Phi(\tau, x)). \quad (23)$$

By using the viscosity notion in HJB theory, one can prove the following result:

(i) The function w is Lipschitz continuous.

(ii) We have:

$$w(s,y) = w_0(s,y) \qquad \forall s \in [s_*^{max}, T],\ \forall y \in R^6. \quad (24)$$

(iii) The function w is the unique Lipschitz continuous viscosity solution of the following HJB equation:

$$\min(-\partial_s w + H(s,y,D_y w), w - d_K(y)) = 0, \text{ on } (s_*^{max}, T) \times R^6 \quad (25a)$$

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$$\min(-\partial_s w + H(s, y, D_y w)\,,\ w - d_K(y), w - Mw_0(s, y)) = 0 \text{ on } (s_*^{min}, s_*^{max}) \times R^6 \tag{25b}$$

$$\min(-\partial_s w + H(s, y, D_y w), w - d_K(y)) = 0 \text{ on } (0, s_*^{min}) \times R^6 \tag{25c}$$

$$w(T, y) = d_C(y) \vee d_K(y) \qquad \text{for } y \in R^6 \tag{25d}$$

Equation (25a) together with the final boundary condition (25d) confirm the assertion that $w \equiv w_0$ on $[s_*^{max}, T] \times R^d$. Also, equation (25b) indicates that the optimal strategy may include at most one switch at time $s_* \in [s_*^{min}, s_*^{max}]$. By definition of the value function w, it follows that:

$$\begin{aligned} &w(0, y) \leq 0 \Leftrightarrow \exists s_* \in [s_*^{min}, s_*^{max}],\ \exists \tau_B \in [0, \tau_B^{max}] \\ &\exists u_1 \text{ on } [0, s_*] \text{ s.t. } w_0\left(s_*, \Phi\left(\tau_B, y_y^{u_1}(s_*)\right)\right) \leq 0 \end{aligned} \tag{26}$$

5. Application of the HJB approach to launcher trajectory optimization

As in [1], a formulation of the launcher trajectory optimization has been proposed in order to replace this high dimension problem by a set of smaller ones, without losing the global optimality. The principle is to consider a cost function which is separable in time, or in other word a cost function which value depends at each time only on the current state (and naturally on the control), and not on the history of previous states. In such a frame, if the problem to optimize relies on control and parameters occurring at different phases, it is possible to cut the global problem in smaller ones only ruled by current "active" control and/or parameters. Each sub problem can be solved sequentially, while the global optimality is maintained thanks to the combining effort of memorization operated at the end of each sub propagation phase and to the matching operated between the sub problems during the global reconstruction phase.

5.1 Resolution procedure for the problem $(\boldsymbol{P})$

Let $y = (x(p), m) \in X_0 \times [0, +\infty[$ be an initial state composed of the physical state $x(p) \in X_0$, state of the launcher at the end of Phase 0 of the flight corresponding to a choice of shooting parameters $p \in P_{ini}$ and payload mass m. From (21) and (24), it follows that if $w(0, y) \leq 0$, then there exists a time $s_* \in [s_*^{min}, s_*^{max}]$, a ballistic time $\tau_B \in [0, \tau_B^{max}]$, and a control law $u(\cdot)$ defined on $[0, T]$ of the form:

$$u(s) = \begin{cases} u_1(s) & \text{if } s \in [0, s_*[\\ \\ u_2(s) & \text{if } s \in [s_*, T[\end{cases} \tag{25}$$

such that the corresponding solution y_y^u satisfies:

$$\begin{cases} y(0) = y \in X_0 \times [0, +\infty[,\ y_y^u(s) \in K\ \forall s \in [0, T] \\ y(s_*^+) = \Phi(\tau_B, y(s_*^-)), \quad \text{and } y_y^u(T) \in C. \end{cases} \tag{26}$$

The HJB equation is set in a whole continuous state domain; nevertheless, in order to perform computations a finite domain is used, on which the HJB equations (25a), (25b), (25c) and (25d) are discretized. For this a uniform space grid is chosen; the time step is chosen constant for simplicity. Advanced numerical technics are then applied to efficiently evaluate the approximation of the function w which corresponds to the propagation phase previously described.

Therefore, to solve the problem (P), one can proceed as follows:
- compute the set X_0 of possible states of the launcher that can be reached at the end of Phase 0 for a large sample of parameters (ψ, ω);

- evaluate the application Φ for different starting points of the grid associated to the first upper stage boost, and for different ballistic phase durations, this parameter being the only one controlling the phase (no engine thrust is available by definition);

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- solve the HJB equation of Theorem 2 to get an approximation of w_0. The computation starts from the GEO orbit, which is perfectly known; then, all the sets of kinematic conditions are time reverse propagated, the control being the orientation of the thrust;

- solve the HJB equation (25) to obtain an approximation of the value function w. To do so, a second HJB retro-propagation is considered from the end of the first upper stage boost to the beginning of the exo-atmospheric phase.

- define on the set X_0 the function:

$$m^*(x) = \sup\{m \mid w(0,(x,m)) \leq 0\} \tag{26}$$

This function corresponds to the biggest payload mass that is possible to steer to the GEO orbit starting from x. Finally, the optimal mass is given by:

$$m_{opt} = \sup_{x \in X_0} m^*(x) \tag{27}$$

- reconstruct the optimal trajectory. Let $x^* \in X_0$ be such that $m^*(x^*) = m_{opt}$. By the definition of the set X_0 one can identify the optimal shooting parameters $p^* \in P_{ini}$ such that $x^* = \Gamma(p^*)$. Hence, we get the shooting parameters $p^* = (\psi^*, \omega^*)$. Moreover, by using the value function w, one can reconstruct an optimal trajectory and get the duration s_* of the first boost of the second engine, the duration τ_B of the ballistic flight, and the control laws of the phases 1, 2 and 4. The optimal trajectory and associated control are finally determined by performing the reconstruction among the trajectories resulting from the matching of the different sub problems.

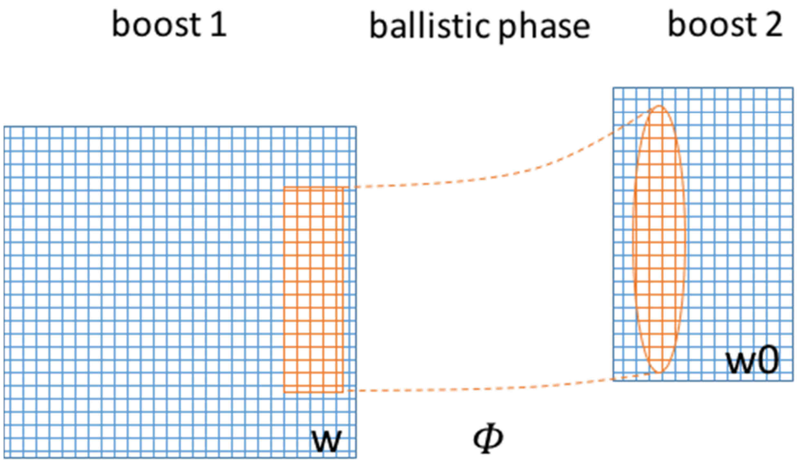

Figure 5: Schematic representation of the numerical implementation for the HJB computations

The dynamic states at the interfaces of all the sub problems are handled thanks to multi-dimensional interpolations minimizing the loss of information occurring at the encounter of the different dynamic state “wave fronts” generated during the propagation phases.

5.2 Numerical results

The launcher trajectory optimization toward GEO is solved by applying the HJB described previously. Let us precise some elements about the flight dynamics considered: the launcher is considered to respond instantaneously to the control (thrust orientation); in particular, inertia dynamics around the center of gravity is considered to be negligible, the launcher is a point on which are concentrated the weight, thrust and aerodynamics forces. The trajectory is considered in three-dimensional space: the state includes three positions, three velocities, and scalar parameters: pitch over rate, launch azimuth, ballistic phase duration, allocation of propellant consumption among the two upper stage boosts, payload mass. This leads to 11 dimensions, to which the time (which is also discretized) should be added.

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The trajectory is optimized with two independent algorithms: one relying on a shooting method, the other corresponding to the HJB implementation strategy previously described. The trajectory optimized by the shooting method is named “referenced trajectory”: the associated numerical code is validated and representative of CNES trajectory/performance studies performed in standard launcher engineering analysis.

It should be recalled that the flight dynamics implemented in HJB algorithm had already been validated in [1], by comparing a trajectory simulated using the flight dynamics implemented in HJB algorithm and taking into account the control associated to the reference trajectory, to the reference trajectory.

Now, an HJB optimization is performed, in order to solve the initial problem presented in §2.1: to maximize the payload mass on the GEO orbit, while optimizing all controls and parameters. Several computations are run, each corresponding to a different state grid discretization:

Table 1: Grid sizes considered

Optimization	**Grid**	**Size (r × L × latitude × v × gamma × Khi)**
1	boost1	20×20×20×20×30×3
1	switch	20×15×20×20×20×3
1	boost 2	20×15×20×20×20×3
1	payload mass range (kg)	[$0.96m_0$ - $1.02m_0$]
2	boost1	20×20×20×20×30×3
2	switch	20×15×20×20×20×3
2	boost 2	20×15×20×20×20×3
2	payload mass range (kg)	[$0.94m_0$ – $1.06m_0$]
3	boost1	30×20×30×30×40×4
3	switch	25×15×30×30×30×4
3	boost 2	25×15×30×30×30×4
3	payload mass range (kg)	[$0.94m_0$ – $1.06m_0$]

The “Optimization 1” grid corresponds to a rude grid allowing to reduce the amount of computation. The only difference between “Optimization 2” grid and the previous one is the payload mass range amplitude. The “optimization 3” grid is refined with regards to the previous one. The associated results at the end of the HJB optimizations are given hereafter:

Table 2: Trajectory characteristics

Data	**PMP**	**HJB**		
	Reference trajectory	**Optimization 1**	**Optimization 2**	**Optimization 3**
Propellant consumption of upper stage first boost (kg)	19 746	19 802	20 138	20 173
Propellant consumption of upper stage second boost (kg)	4 754	4 545	4 291	4 320
Payload mass (kg)	m_0	m_0	m_0	$1.01m_0$
Duration of the ballistic phase (s)	18 885	18 609	19 075	18 706

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The comparison of the "reference" and "optimization 1" trajectories shows that the HJB computation offers a first good approximation of the optimal trajectory with a coarse state grid discretization, allowing to get a preliminary result in a short time at limited computation cost.

Compared to "optimization 1" trajectory, the "optimization 2" trajectory allows to verify that the value of the maximized criterion (payload mass) is non sensitive to the payload mass range amplitude, which contributes to validate the overall numerical stability of the HJB computation. The differences on the propellant consumptions between boosts 1 and 2, and on the ballistic phase duration between the two boosts, are moderate, and reflects the flatness of the optimization problem for this phase of flight.

With respect to "optimization 2" trajectory, the "optimization 3" trajectory shows a slight improvement in terms of payload mass, associated to limited variations of propellant consumption repartition and ballistic phase duration. As preliminary characterized in [1], the HJB computations relies partly on the grid fineness, which is adjusted as a compromise between numerical accuracy and computation time. In the sense, the moderate differences with respect to the PMP optimal solution should also be relativized considering the analysis of the accuracy on injection conditions discussed hereafter.

The trajectory profile of the "reference" and "Optimization 3" trajectories are plotted from the launch until the end of the first boost of the upper stage here below.

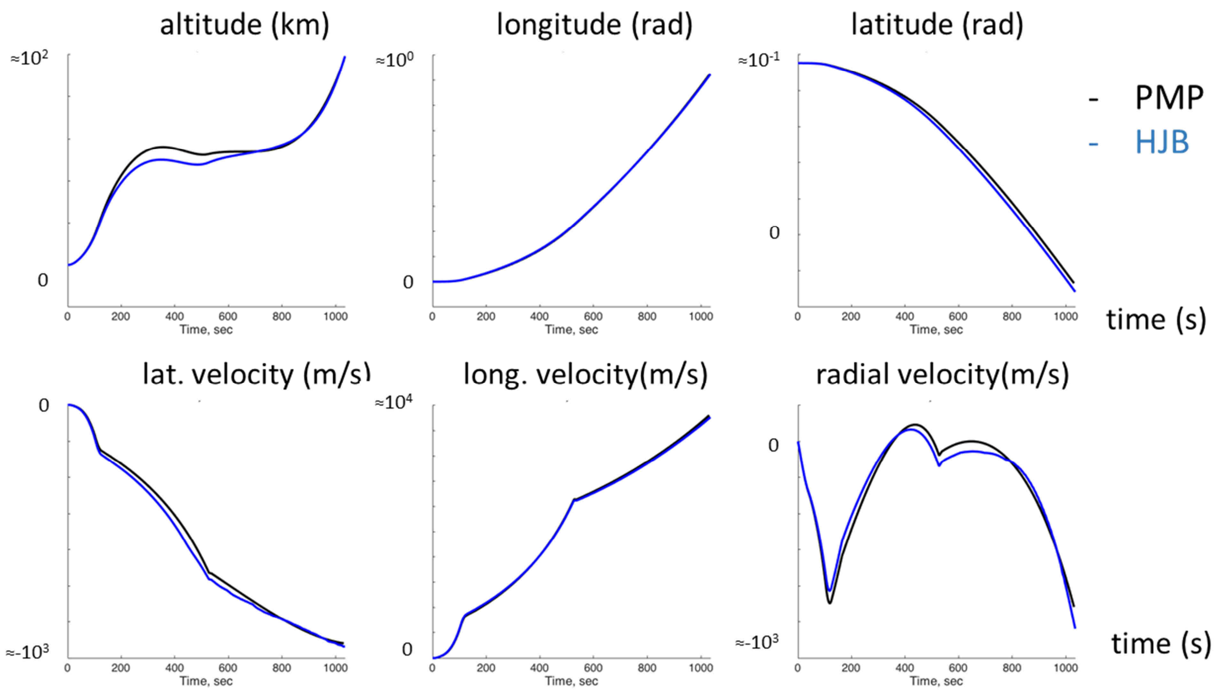

Figure 6: Opt3 trajectory profile – boost1

These results are completed with the second boost of the upper stage, presented on the next page.

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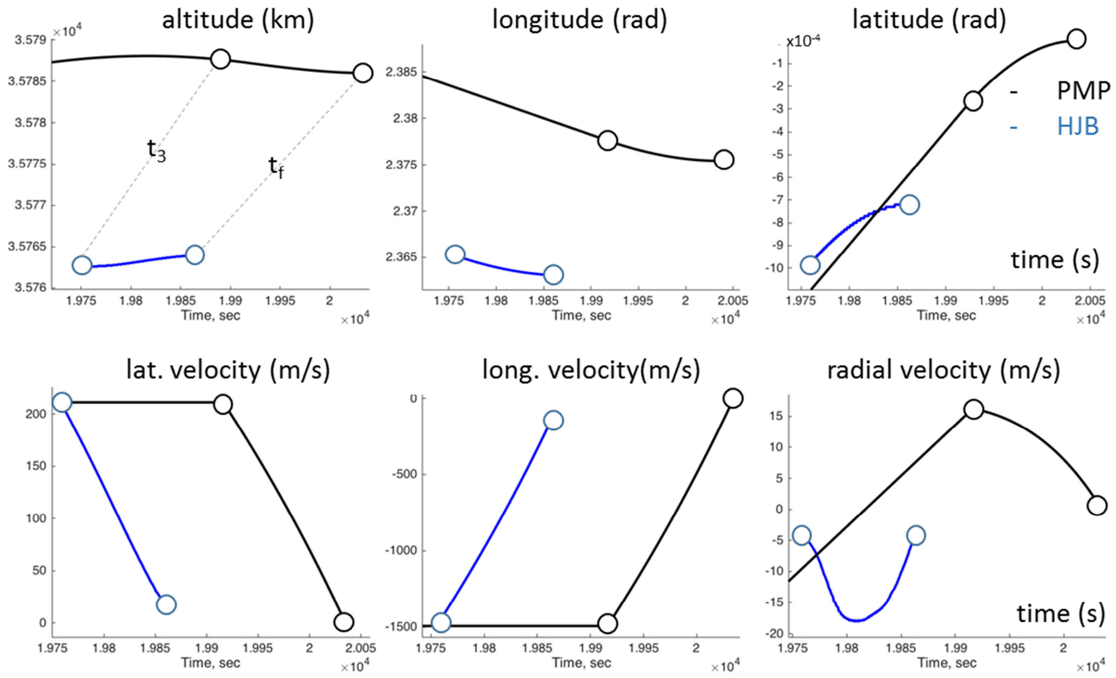

Figure 7: Opt3 trajectory profile – boost2

The “reference” and “optimization 3” trajectories have a global good consistence, which is even increased with respect to the results shown in [1]. As recalled in §2, the maximal payload mass was determined by solving a sequence of HJB problems, the number of the latter being limited so as to maintain a reasonable total computation time. The fact that the payload mass is now explicitly managed in the HJB computation helps to determine the payload mass more accurately. Moreover, it should be recalled that the second upper stage boost used to be considered as an impulsive one in [1], corresponding to forget the trajectory losses occurring during this maneuver. Now, the whole launcher orbital transfer is fully modelled in the HJB computations, which helps to improve the computation accuracy.

As shown in Figure 7, the altitude at injection suffers from a moderate inaccuracy; the velocity, which is here expressed relatively to the target (GEO orbit), is not totally nulled at the injection. These errors may probably be related to the grid discretization and could be solved by refining it.

The computation time is roughly five hours, it is almost divided by two with respect to the previous analysis, where several HJB computations were required to find the optimal trajectory. It is worth noting that the implementation relies on parallel computation, taking benefit of the power offered by a computer network organized through a cluster. While still being a non-negligible amount of time, it should be emphasized that the process is fully automatic, freeing engineers from time consuming effort such as initialization and convergence tasks. Moreover, this duration is small enough to allow computation in “bash mode” during a single day work, or during non-working hours leading to results available at the beginning of next working session.

Hereafters are given some illustrations of the trajectory profile, from launch base to final GEO orbit (in red are the propelled phases, ballistic phases are in blue):

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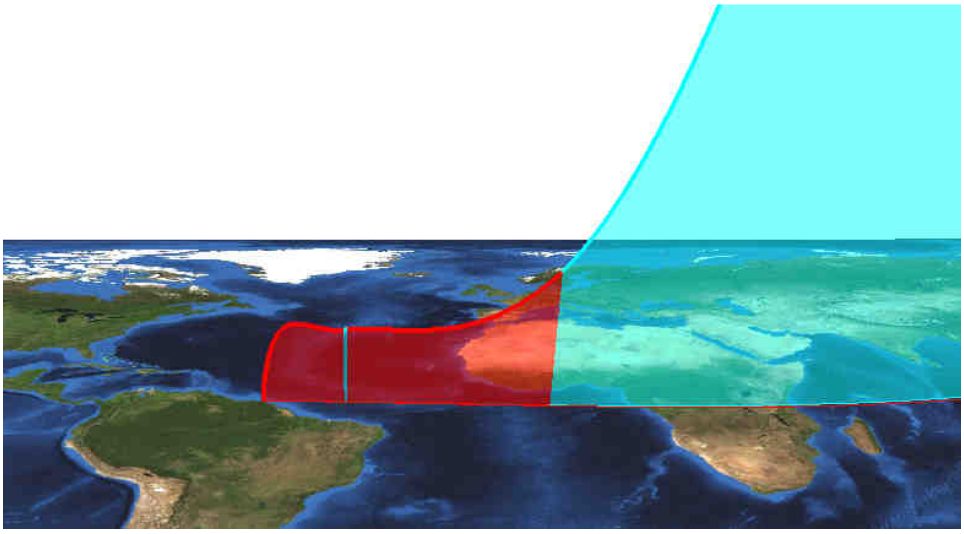

Figure 7: Trajectory profile from launch until the upper stage first boost cut-off

Figure 8: Trajectory profile of the upper stage second boost

6. Perspectives

An improvement axis would be to consider the HJB approach as a way to improve local indirect approach: indeed, the gradient of the level set function corresponds to the dual state considered in the Pontryagin Maximum Principle. The HJB approach could be used to provide a correct estimate of the dual state at the beginning of the trajectory, feeding the shooting method classically derived from the expression of transversality relationships. The HJB approach would be used with a not too much refined grid to speed up calculations; then the local approach would only be used to converge quickly toward the global optimum.

Acronyms

HJB	Hamilton Jacobi Bellman
PMP	Pontryagin Maximum Principle
CNES	Centre National d'Etudes Spatiales
ENSTA	Ecole Nationale Supérieure des Techniques Avancées
UMA	Unité des Mathématiques Appliquées
GTO	Geostationary Earth Orbit
GEO	Geostationary Transfer Orbit
Lsc	Lipschitz semi continuous

Nomenclature

t	Time
t_0	Lift-off time, in seconds
t_1	Start of first upper stage boost
t_2	End of first upper stage boost
t_3	Start of second upper stage boost
t_f	End of second upper stage boost (injection on GEO)
r	Geocentric altitude, in kilometers
L	Longitude with respect to Greenwich meridian, in radians

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l	Latitude with respect to Equator, in radians
v	Launcher velocity magnitude, in meter per second
γ	Launcher velocity slope with respect to local horizon, in radians
χ	Launcher velocity azimuth with respect to instantaneous trajectory plane, in radians
f(.)	Launcher dynamics
X_0	Set of initial state
C	State target
y_{ux}	A trajectory satisfying the flight dynamics
U	Set of admissible controls
K	Set of admissible trajectories
ψ	tilting angle rate
ω	launch azimuth
τ_B	Duration of the ballistic phase
t_f	Final time (injection on GEO)
s	Combustion time
m	Payload mass
m_0	Payload mass injected on GEO orbit for the PMP reference trajectory
x	Kinematic state
y	Kinematic + mass state

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