NUMERICAL IMPEDANCE MATRIX : APPLICATION TO THE HERSCHEL-QUINCKE TUBE CONCEPT

Romain Maréchal*, Emmanuel Perrey-Debain, Jean-Michel Ville, Benoit Nennig

Laboratoire Roberval, Université Technologique de Compiègne,
60200 Compiègne, FRANCE
*E-mail address : romain.marechal@utc.fr

Abstract

The Herschel-Quincke (HQ) tubes, consisting in putting tubes in derivation along a main wave guide, are used as passive devices to control fan noise. In order to assess the efficiency of this system, analytical and semi-analytical models proposed in the literature all rely on the assumption that only plane waves are allowed to propagate in the tube which then restrict the pressure and the normal acoustic velocity to be constant over the duct-tube interface. We show that these simplifications are too limited especially when dealing with high frequency modes in the main duct. To make some progress, a new mixed analytical-numerical approach is proposed. The method, based on the discrete modal analysis of the tube allows to take into account its exact shape as well as the non-uniformity of the acoustic velocity at the interface.

1 INTRODUCTION

One of the most significant sources of noise of an aircraft is due to the propelling system. This noise, which is present during all flight phases, can be decomposed into several types : classical jet noise outside the exhaust nozzle and inner turbo machinery noise (fan, compressor, turbine & combustion). In particular, fan noise is responsible for pure tones at the Blade Passage Frequency (BPF) harmonics, due to the interaction between the rotor wakes and the stator vanes. In order to reduce noise level in modern turbofan engines, sound waves generated by the fan are typically absorbed by acoustic lining covering the duct engine. Though efficient, these treatments seem to have reach their limit and there is still a need for considering other passive techniques to reduce further the sound radiation from the duct outlet. In this context Herschel-Quincke tubes concept could prove to be a reliable option.

In 1833, Herschel [6] first discussed the idea of using acoustic interferences of tones by simply connecting a tube to the main duct in view of reducing the transmitted acoustic waves. 33 years later, Quincke [9] experimentally validated Herschel's theory and many works and experiments have been carried out to explain physical phenomena and explore the potentiality of this system as a noise control device [2, 13].

The assessment of the efficiency of such a system requires a precise knowledge of the acoustic field in the duct. Though standard Finite Element (FE) software could, in principle, be used for this purpose, a full 3D FE model would be extremely demanding as the number of variables is expected to grow like f^3 (f is the frequency). This can have a negative impact when, for instance, some efficient optimizations (geometry of the HQ tubes and their positions) are needed.

Assuming plane wave propagation, the resonance behavior of two duct combination was first established analytically by Selamet *et al.* [11] and then extended to a multiple duct configuration [12]. The proposed approach is simple to implement and allows a very fast computation of the transmitted wave but it is unfortunately limited to low frequency applications. To make some progress, Brady [1] proposed a two-dimensional model including multi-modal analysis in the main duct using a Green's function formalism. The three-dimensional model was then extended by Hallez [4]. Finally, Poirier [8] proposed an improvement by taking into account the exact shape of the interface between the main cylindrical duct and the HQ tube. All the authors just cited simplified their analysis by assuming that the acoustic velocity is *constant* over the duct-tube interface. Furthermore they all modelized the HQ tubes as if they were straight waveguides in which only plane waves are allowed to propagate.

Because these assumptions are known to break down as the frequency increases (see for instance Tang & Lam [14]), the aim of the present work is to propose a model taking into account (i) the exact shape of the HQ tube(s) and (ii) the non uniformity of the acoustic velocity over the interfaces. We will show that these improvements can be made with a relatively small additional computational cost while leading to very accurate results even in the mid-frequency regime.

2 PROBLEM STATEMENT

The problem under consideration as illustrated in Fig. 1 consists of a two-dimensional main duct Ω of width h on witch is connected a single HQ tube. Given an incident pressure wave P_i stemming from the left, we wish to evaluate the reflected wave P_r as well as the transmitted one, P_t. We call Γ_w the rigid wall of the main duct and Γ_{HQ} the interfaces with the HQ tube.

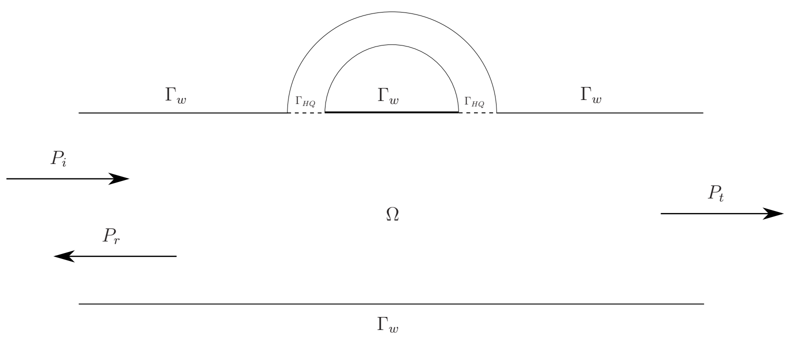


Figure 1: Main duct with one Herschel-Quincke tube

In the main duct and in the HQ tube, the acoustic pressure p must satisfy the Helmholtz equation

$$\Delta p + k^2 p = 0 \tag{1}$$

as well as the hard wall condition on Γ_w, that is $\partial_n p = \mathrm{i}\omega\rho v_n = 0$. Here, we adopt $e^{-\mathrm{i}\omega t}$-convention, $k = \omega/c$ is the wave number and v_n denotes the acoustic velocity normal to the boundary. Finally, we require that p and its normal derivative (or v_n) to be continuous across the interface Γ_{HQ}.

3 MIXED NUMERICAL-ANALYTICAL MODEL

In this section, we shall present the main ingredients of the method, that is (i) the establishment of a numerical impedance matrix describing the dependence of p and its normal derivative on both interfaces of the HQ tube (ii) the Green's formalism in the main duct.

3.1 *Calculation of the numerical impedance matrix*

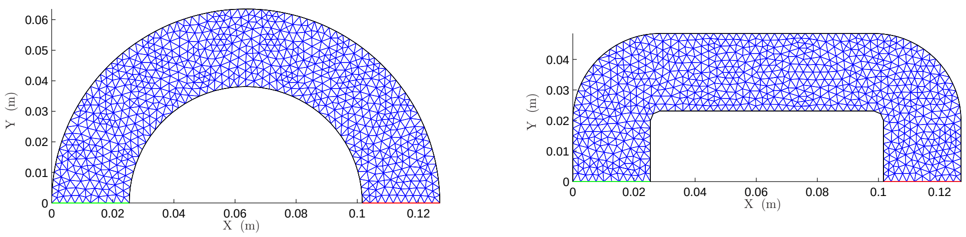
Impedance matrices considered in Ref. [1, 4, 11, 12] are built with the restriction that only plane waves are allowed to propagate in the HQ tube. By calling L the average HQ tube length (see Fig. 3), the 2×2 matrix has the explicit form

$$\mathbf{Z}(\omega) = \frac{1}{k\sin(kL)} \begin{bmatrix} \cos(kL) & 1 \\ 1 & \cos(kL) \end{bmatrix} \tag{2}$$

where it is understood that the pressure and its normal derivative at the interface are connected via the impedance condition

$$\mathbf{p}_{int} = \mathbf{Z}(\omega) \left. \frac{\partial \mathbf{p}}{\partial n} \right|_{int}. \tag{3}$$

Under the plane wave assumption, the vector $\mathbf{p}_{int}$ simply contains the value of the constant pressure on both interface and similarly for the normal derivative.



(a) (b)

Figure 2: Meshing of different HQ tubes with linear triangular elements : (a) half-circular (805 nodes) and (b) “drainpipe-shaped” (826 nodes) tubes

In order to take into account the acoustic particle velocity variation at the interfaces, numerical impedance matrices can be built via finite element discretization of the acoustic pressure field in the tube HQ. For the sake of illustration, typical meshes for two different tube shapes are shown in Fig. 2. The first step is to compute a set of eigenmodes ($\mathbf{\Phi}_n$, ω_n) of the tube with rigid wall conditions on the boundary. Once the number N_m of modes has been chosen, a numerical impedance matrix can be recovered as follows

$$\mathbf{Z}(\omega) = \langle \tilde{\mathbf{\Phi}}_1 \dots \tilde{\mathbf{\Phi}}_{N_m} \rangle \begin{bmatrix} \frac{1}{\omega_1^2 - \omega^2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{\omega_{N_m}^2 - \omega^2} \end{bmatrix} \langle \tilde{\mathbf{\Phi}}_1 \dots \tilde{\mathbf{\Phi}}_{N_m} \rangle^{\mathrm{T}} + \mathbf{R}(\omega) \quad (4)$$

Here, the tilde symbol signifies that we only retain the nodal values of the eigenmodes on the interface. Similarly, the vector $\mathbf{p}_{int}$ (resp. $\partial_n \mathbf{p}|_{int}$) now contains the value of the pressure (resp. the normal derivative) at each node of the FEM mesh on the interface. The interest of such a decomposition is that when the frequency of interest is taken well below the highest modal resonant frequency (i.e. $\omega \ll \omega_{N_m}$), the correction term $\mathbf{R}(\omega)$ is quasi-constant so we can take $\mathbf{R}(\omega) \approx \mathbf{R}(0)$ and store the so-called static correction matrix once for all. This renders the computation of (4) like a very fast and simple procedure.

In the present study, the maximum frequency of interest is f_{max} = 5000 Hz so we choose to truncate the modal basis at $4f_{max}$= 20000 Hz. In this scenario only the first 350 eigenmodes are kept for the calculation of the numerical impedance matrix.

3.2 *Green's formalism in the main duct*

The theory starts by introducing the hard-walled duct Green's function satisfying the usual modal radiation condition on both ends of the main duct, i.e.

$$G(\mathbf{x}, \mathbf{x}_0) = -\sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(x_0)}{2\mathrm{i}\beta_n} e^{\mathrm{i}\beta_n |z - z_0|} \quad (5)$$

where $\mathbf{x} = (x, z)$ and $\mathbf{x}_0 = (x_0, z_0)$ are two points in the propagative domain Ω. Functions ψ_n are the transverse modes satisfying the hard-wall conditions

$$\psi_n(x) = \begin{cases} \dfrac{1}{\sqrt{h}} & \text{if } n = 0, \\ \sqrt{\dfrac{2}{h}} \cos(\alpha_n x) & \text{otherwise.} \end{cases} \quad (6)$$

where $\alpha_n = n\pi/h$ and β_n are the associated axial wave numbers given by the dispersion relation as

$$\beta_n = \sqrt{k^2 - \alpha_n^2}. \quad (7)$$

Using the Green's theorem, the pressure anywhere in the duct is given via the convolution integral

$$p(\mathbf{x}) = P_i(\mathbf{x}) + \int_{\Gamma_{HQ}} G(\mathbf{x}, \mathbf{x}_0) \frac{\partial p}{\partial n}(\mathbf{x}_0) \, \mathrm{d}\Gamma(\mathbf{x}_0) \tag{8}$$

The integral over Γ_{HQ} is then computed using the same discretization scheme as for the boundary nodes of the HQ tube FE mesh. If we call N_{elt} the total number of finite elements on the boundary (Γ_i for $i = 1, \ldots, N_{elt}$), then assuming standard piecewise linear interpolation, we have

$$\begin{aligned}\int_{\Gamma_{HQ}} G(\mathbf{x}, \mathbf{x}_0) \frac{\partial p}{\partial n}(\mathbf{x_0}) \, \mathrm{d}\Gamma(\mathbf{x}_0) = \sum_{i=1}^{N_{elt}} \int_{\Gamma_i} & G(\mathbf{x}, \mathbf{x}_0) \\ & \times \left(\left.\frac{\partial p}{\partial n}\right|_{i,1} N_1(\mathbf{x}_0) + \left.\frac{\partial p}{\partial n}\right|_{i,2} N_2(\mathbf{x}_0) \right) \mathrm{d}\Gamma_i(\mathbf{x}_0)\end{aligned} \tag{9}$$

where N_1 and N_2 stand for the standard linear shape functions and $\partial_n p|_{i,1}$ and $\partial_n p|_{i,2}$ are the nodal values of the normal derivative of the pressure on the i^{th} element.

Finally by collocating the integral equation precisely on each node of the problem, we find the discrete form of (8) as

$$\mathbf{p}_{int} = \mathbf{P}_i + \mathbf{G}(\omega) \left.\frac{\partial \mathbf{p}}{\partial n}\right|_{int} \tag{10}$$

where the frequency dependence of the Green matrix $\mathbf{G}$ has been highlighted on purpose.

Physically, the Green matrix can be interpreted as the impedance matrix of the main duct.

3.3 *Matrix system for several tubes*

The same procedure can be repeated for an arbitrary number N_{tubes} of HQ tubes. This then yields the general block diagonal form

$$\mathbf{p}_{int} = \underbrace{\begin{bmatrix} \mathbf{Z}_1(\omega) & & 0 \\ & \ddots & \\ 0 & & \mathbf{Z}_{N_{tubes}}(\omega) \end{bmatrix}}_{\mathbf{Z}(\omega)} \left.\frac{\partial \mathbf{p}}{\partial n}\right|_{int} \tag{11}$$

where $\mathbf{Z}_n$ is the numerical impedance matrix of the n^{th} tube. Now using the previous result together with (10) yields the numerical solution for the acoustic velocity vector at the interface as

$$\left.\frac{\partial \mathbf{p}}{\partial n}\right|_{int} = (\mathbf{Z}(\omega) - \mathbf{G}(\omega))^{-1} \mathbf{P}_i \tag{12}$$

The reflected and transmitted pressures waves can then be calculated using (8). Because the method just described is based on the analytical solution of the Green's function and on the numerical discretization of the tubes, this shall be referred as the mixed numerical-analytical model as opposed to the full FE model of the problem which is then purely numerical.

4 RESULTS AND CONCLUDING REMARKS

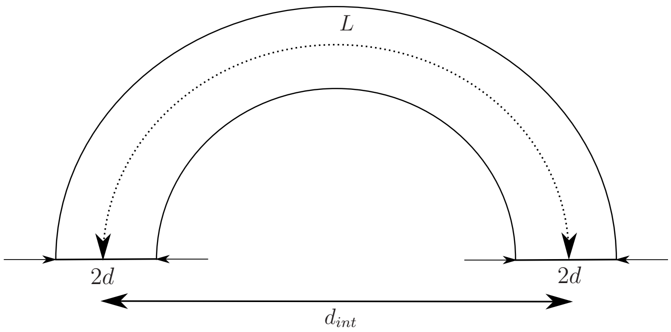


Figure 3: Parameters of a HQ tube

This section presents numerical results based on the HQ tube displayed in Fig. 3 which dimensions can be found in Ref. [11]. These are recalled in Table 1

Radius	$d = 0.02337$ m
Distance between interfaces	$d_{int} = 0.3985$ m
Average length of HQ tube	$L = 0.7845$ m

Table 1: Parameters of the HQ tube

The width of main duct is $h = 2a = 0.04859$ m. The results are presented with respect to the dimensionless variable ka. The study is carried out from very low frequency up to $ka_{max} = 2.23$ (which corresponds to $f_{max} = 5000$ Hz) with a very small stepsize 1 Hz. In the overall frequency range, the incident pressure is a plane wave wave and above cut-off (see Table 2), the next order mode becomes propagative.

Mode number	Cut-off frequency
0 (plane)	$ka = 0$
1	$ka = \frac{\pi}{2}$ ($f \approx 3523$ Hz)

Table 2: Propagative modes in the main duct

In Fig. 4 are plotted the Transmission Loss (TL) curves computed with three different methods. The TL is defined as the ratio of transmitted acoustic power with respect to the incident one, that is

$$\text{TL} = 10\log_{10}\left(\frac{1}{\beta_0\left|A_0^i\right|^2}\sum_{n=0}^{M_{prop}-1}\beta_n\left|A_n^t\right|^2\right). \tag{13}$$

Here, M_{prop} is the number of propagative modes, A_0^i is the amplitude of the incident plane wave and A_n^t is the transmitted modal amplitude of the n^{th} mode.

The black curve in Fig. 4 is related to the mixed model. In this example, each interface of the HQ tube contains about 12 elements.

On the left, we compare the results with those obtained with standard FE method using 28684 degrees of freedom. In this latter, the non reflecting conditions are imposed using the Dirichlet-to-Neumann (DtN) map as described for instance in Refs. [3, 5, 10]. The very good agreement validate our method and the small discrepancies noticeable at high frequency are thought to be due to the full FE model which then start losing accuracy.

The green curve on the right is the TL obtained with the plane-wave model of Brady [1]. This shows noticeable differences as the frequency increases and this would have been even more remarkable by considering the next propagative mode in the incident field. It is already anticipated, from other calculations not shown here, that the differences should be even amplified in the 3D case.

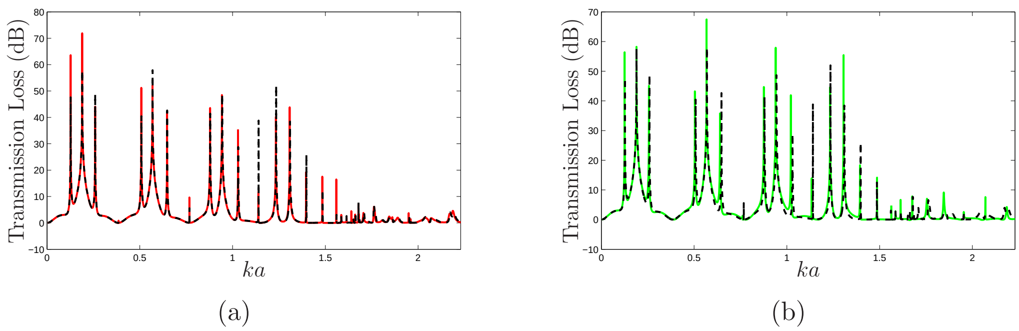


Figure 4: Half-circular HQ tube transmission loss : —— full finite element model, —— model [1] with plane wave in straight HQ tube, – – mixed model

Fig. 5 shows the importance of taking into account the real geometry of the HQ tube. In this example, both HQ tubes have the same parameters (see Table 1) and only the geometry is different (see Fig. 2). If the circular pipe is expected to behave almost like a straight waveguide (at least in the low frequency regime), the small curvature of the drainpipe-shaped pipe can have noticeable effects on the TL due to some internal reflection of the sound waves in the pipe and this is clearly visible on the graph especially above the first cut-on frequency.

It is instructive to observe that the first series of peaks corresponds to two types of destructive interferences as discussed in [7, 8, 11]. The *type I* interference occurs when the acoustic pressure is zero at the downstream interface but nonzero at the upstream interface. The *type II* interference occurs when the acoustic pressure is zero

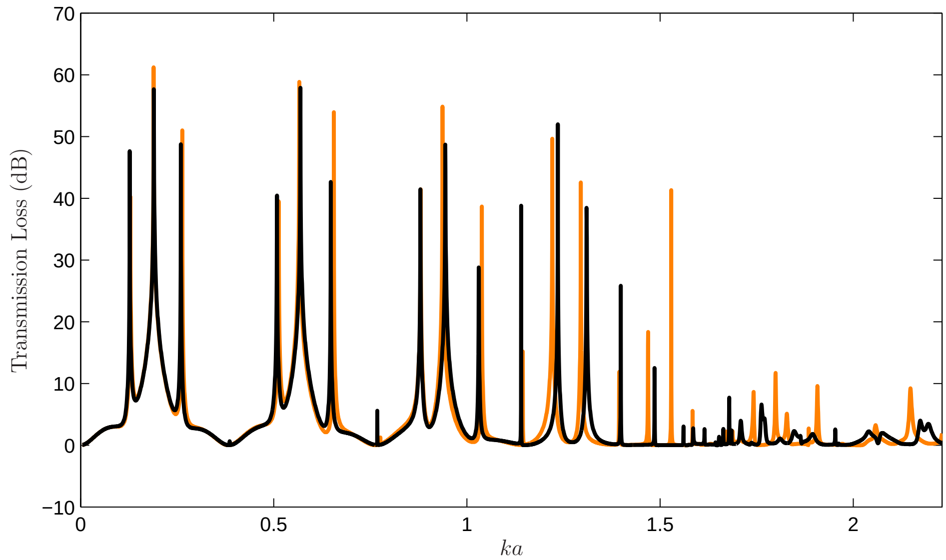


Figure 5: Effect of the HQ tube shape on the Transmission Loss: — "drainpipe-shaped" HQ tube and — circular HQ tube.

at both interfaces. These resonances, evaluated from the plane-wave model, are given explicitly by

$$\begin{cases} (L - d_{int}) = (2n-1)\,\dfrac{\lambda}{2} & \textit{type I}, \\ (L + d_{int}) = m\lambda & \textit{type II}. \end{cases} \tag{14}$$

where λ is the 'resonant' wavelength and $m, n \in \mathbb{N}^*$. It is found in this example that these formulas only hold for sufficiently low frequency (say $ka < 1$).

From these early results, it has been shown that the real geometry of the HQ tubes and the non-uniformity of the normal acoustic velocity at the interfaces can have noticeable effects on the TL. Keeping in mind that typical frequencies of interest in turbofan engines can reach as much as $ka \approx 50$ in a cylindrical duct (a is the radius), it is anticipated that the proposed mixed-numerical model should prove to be extremely beneficial in designing optimized HQ systems.

References

[1] L.A. Brady. Application of the herschel-quincke tube concept to higher-order acoustic modes in two-dimensional ducts. Master's thesis, VPI (Blacksburg, Virginia), USA, 2002.

[2] R.A. Burdisso and W.F. Ng. Fan noise control using herschel-quincke resonators. *NASA Report*, CR-2003-212097, 2003.

[3] D. Giljohann and M. Bittner. The three-dimensional dtn finite element method for radiation problems of the helmholtz equation. *J. Sound Vib.*, 212(3):383–394, 1998.

[4] R.F. Hallez. Investigation of the herschel-quincke tube concept as a noise control device for turbofan engines. Master's thesis, VPI (Blacksburg, Virginia), USA, 2001.

[5] I. Harari, I. Patlashenko, and D. Givoli. Dirichlet-to-neumann maps for unbounded wave guides. *J. Comput. Phys.*, 143(1):200–223, 1998.

[6] J.F.W. Herschel. On the absorption of light by colored media, viewed in connnection with the undulatory theory. *Philos. Mag.*, 3:401–412, 1833.

[7] M. Karlsson, G. Ragnar, and M. Abom. The herschel-quincke tube : The attenuation conditions and their sensitivity to mean flow. *J. Acoust. Soc. Am.*, 124(2):723–732, 2008.

[8] B. Poirier. *Développement d'un modèle analytique opur le calcul de la matrice de diffusion d'un tronçon composé de tubes HQ*. PhD thesis, UTC (Compiègne), France, 2008.

[9] G. Quincke. Ueber interferenzapparate für schallwellen. *Ann. Phys.*, 204(6):177–192, 1866.

[10] S.P. Sari. *Propagation acoustique dans des conduits à parois traitées en présence d'écoulement : modélisation par la méthode des éléments finis*. PhD thesis, Université de Bourgogne (Dijon), France, 2008.

[11] A. Selamet and N.S. Dickey. The herschel-quincke tube: A theoretical, computational, and experimental investigation. *J. Acoust. Soc. Am.*, 96(5):3177–3185, 1994.

[12] A. Selamet and V. Easwaran. Modified herschel-quincke tube : Attenuation and resonance for n-duct configuration. *J. Acoust. Soc. Am.*, 102(1):164–169, 1997.

[13] J.P. Smith and R.A. Burdisso. Experimental investigation of the herschel-quincke tube concept on the honeywell tfe731-60. *NASA Report*, CR-2002-211431, 2002.

[14] S.K. Tang and G.C.Y. Lam. On sound propagation from a slanted side branch into an infinitely long rectangular duct. *J. Acoust. Soc. Am.*, 124(4):1921–1929, 2008.