Global stability analysis – a key enabler in ROM and flow control

Marek Morzyński[1], Witold Stankiewicz[1], Frank Thiele[2], Bernd R. Noack[2] and Gilead Tadmor[3]

[1]*Poznań University of Technology, Institute of Combustion Engines and Transportation, Poland*

[2]*Berlin Institute of Technology, Department of Fluid Dynamics and Engineering Acoustics, Germany*

[3]*Department of Electrical and Computer Engineering, Northeastern University, Boston, USA*

Abstract

In the current study, Reduced Order Models (ROMs) targeting strategies for experimental feedback flow control are disscussed. For practical reasons, that model should incorporate a range of flow operating conditions with small number of degrees of freedom. Standard POD Galerkin models are challenged by the over-optimization at one operating condition (Deane et al. [1]). The extention of dynamic range with additional global flow stability modes is the first applied technique. Further side constraints for control-oriented ROMs are taken into account by a 'least-dimensional' Galerkin approximation based on a novel technique for continuous mode interpolation (Morzyński et al. [2]). This interpolation allows to preserve the model dimension of a single state while covering several states by adjusting (interpolated) modes. The resulting 3-dimensional Galerkin model is presented for the transient flow around NACA 0012 airfoil and shown to be in good agreement with the corresponding direct numerical simulation.

1 Introduction

The computational fluid dynamics (CFD) is a mature tool used in design and improvement of performance of the transport systems like airplanes, trains or cars. Development of High Fidelity CFD systems is accompanied by the clear conclusion that parametric studies cannot relay only on increasing computer CPU power or even parallelization [3]. The design process opens myriads of versions to be analyzed. Low Fidelity Analysis and/or Reduced Order Models (ROMs) are presently the only realistic alternative. The ROMs are also necessary in the close-loop flow control. Model-based flow control requires online-capable feedback laws. In this paper we focus on a system reduction and the use of global flow stability eigenmodes as the key strategy to improve the flow model dynamics. Traditionally, POD modes, being the result of pure signal processing, were used for ROMs. Poor performance of models build with this basis trigered several novel ideas and improvements [1, 4, 5, 6, 7, 8]. The succesfull approaches incorporated more physical information about the modeled system. The use of stability eigenmodes and continuous mode interpolation presented in this paper is the example of this strategy.

2 Empirical Galerkin model

Standard Galerkin method [9] decomposes the velocity field in a base flow $\mathbf{u_0}$ and fluctuation $\mathbf{u}'$. Velocity field is approximated in physical domain Ω with space dependent expansion modes $\mathbf{u_i}$ and time-dependent Fourier coefficients a_j

$$\mathbf{u}^{[0..N]} = \mathbf{u}_0 + \sum_{j=1}^{N} \alpha_j \mathbf{u}_j, \qquad a_0 \equiv 1. \tag{1}$$

The ansatz (1) can serve for deriving high dimensional FEM model (computational Galerkin method) if expansion modes have local compact support on grid cells (FEM's hats). Low dimensionality and robustness which is our goal in designing flow model requires traditional Galerkin method which is based on global expansion modes. The Galerkin system, resulting from projection of the Navier-Stokes equation onto the space spanned by the expansion modes has the form:

$$\frac{d}{dt} a_i = \frac{1}{Re} \sum_{j=0}^{N} l_{ij} a_j + \sum_{j,k=0}^{N} q_{ijk} a_j a_k, \tag{2}$$

where:

$$l_{ij} = \Big(\mathbf{u}_i, \triangle \mathbf{u}_j\Big)_\Omega \quad \text{and} \quad q_{ijk} = -\Big(\mathbf{u}_i, \nabla\cdot(\mathbf{u}_j \otimes \mathbf{u}_k)\Big)_\Omega \tag{3}$$

The pressure term may be neglected in the case of absolutely unstable wake flows and arbitrarily large domains [10]. Equation 2 is a low dimensional analogon of DNS. The RANS-equivalent form together with Finite Time Thermodynamic (FTT) closure is described in details in [11].

3 Global flow stability eigenmodes

The incompressible Navier-Stokes equation

$$\dot{u}_i + u_{i,j} u_j + p_{,i} - \frac{1}{Re} u_{i,jj} = 0 \tag{4}$$

linearized for small disturbances, with the exponential anzatz for the time dependence yields the generalized complex eigenvalue problem:

$$\begin{aligned} \lambda \tilde{u}_i + \tilde{u}_j \bar{u}_{i,j} + \bar{u}_j \tilde{u}_{i,j} + \tilde{p}_{,i} - \frac{1}{Re} \tilde{u}_{i,jj} &= 0 \\ \tilde{u}_{i,i} &= 0 \end{aligned} \tag{5}$$

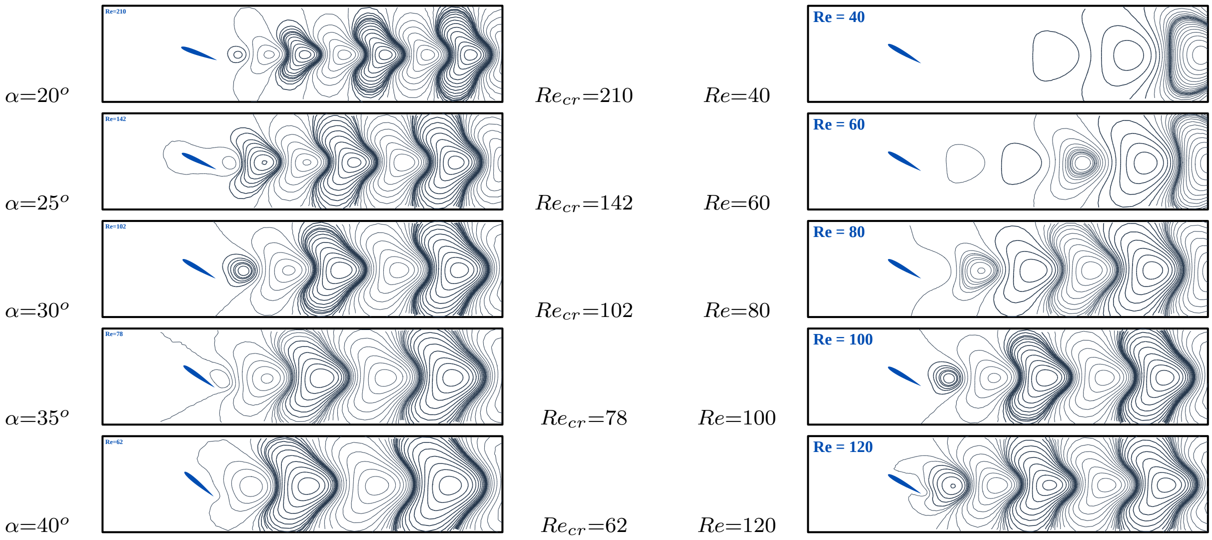

Figure 1: Left: Streamlines of the most unstable modes corresponding to critical Reynolds numbers at different angles of attack. Right: The evolution of the most unstable eigenmodes with the growth of Reynolds number for $\alpha = 30^0$. On the figure, the streamlines are visualised

After discretization Eq. (5) takes the form:

$$Ax - \lambda Bx = 0 \tag{6}$$

characterized by a very large dimension of the weak conditioned unsymmetric matrices. Several papers deal with the solution global flow stability problem to mention for example the recent ones [12, 13, 15] and the review given in [14]. In [16] the solution of large global stability eigenvalue problem with unstructured 3D FEM is disscussed.

Stability analysis is traditionally a tool for prediction of amplification or damping of external disturbances present in all real flows. Usually this method delivers two kinds of information - the physical modes (eigenmodes) being the form of (spatial) disturbance development and eigenvalues where real and imaginary part is the measure of periodicity in time and amplification of a flow structure - growth rate.

The physical modes are of particular interest for flow modeling. In Fig. 1 the real part of the eigenvector is shown for the flow around NACA 0012 airfoil at different flow conditions.

4 Improvements of Galerkin model

4.1 Mean-field correction

Reduced Order Model obtained with the POD Galerkin method is highly efficient and resolves nearly perfectly the kinematics of the flow. At the same time it is highly fragile and sensitive to changes in the parameters or operating conditions. First two POD modes capture about 95% of the perturbation energy, yet Galerkin model, based on these two modes, is structurally unstable.

The inclusion of eight POD modes, capturing the first four harmonics of the attractor, suffices to achieve nearly perfect resolution and structurally stable GS. Yet the correct prediction of dynamic of the system with this model is limited to a small neighborhood of the attractor, and to relatively small Reynolds number perturbations. Stabilization of the GM can be obtained with the shift mode [17] as suggested by mean-field theory. Shift mode is a normalized difference $\mathbf{u}_0 - \mathbf{u}_s$ where $\mathbf{u}_0$ is mean flow solution and $\mathbf{u}_s$ is (unstable) steady solution.

The inclusion of the shift mode reduces model sensitivity to parameter variations and is an enabler for the low dimensional representation of transient manifolds, such as the one connecting the unstable steady solution to the attractor. The dynamics of the minimum Galerkin Model with a shift mode is compared with DNS in Fig. 2. Shift mode is the key enabler for construction of transient, control-oriented models.

4.2 Hybrid model employing stability modes

Further improvement of the model dynamics is obtained with hybrid model employing stability modes [18]. In this model POD resolve the attractor and stability eigenmodes resolve the linearized dynamics. Thus, dynamic transient and post-transient flow behavior was accurately predicted. The concept of hybrid model reduces significantly the number of necessary degrees of freedom of the system. This approach is demonstrated for benchmark problem of the flow around circular cylinder in [17]. The transients of the hybrid models are compared with DNS in Fig. 2. The hybrid model combines the advantages of both reduced models. It converges to the limit cycle preserving initially the growth rate predicted by global stability analysis.

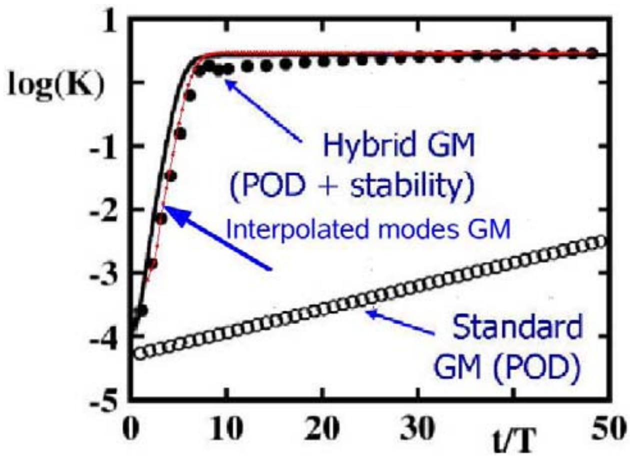

Figure 2: Transients of Galerkin models for flow around a circular cylinder. Improvement - hybrid model and continuous mode interpolation

4.3 Continuous mode interpolation

Furher improvement in designing least-dimensional ROM flow model is continuous mode interpolation technique applied for circular cylinder flow in [2]. The mode interpolation smoothly connects not only different operating conditions, but also stability and POD modes (Fig. 3).

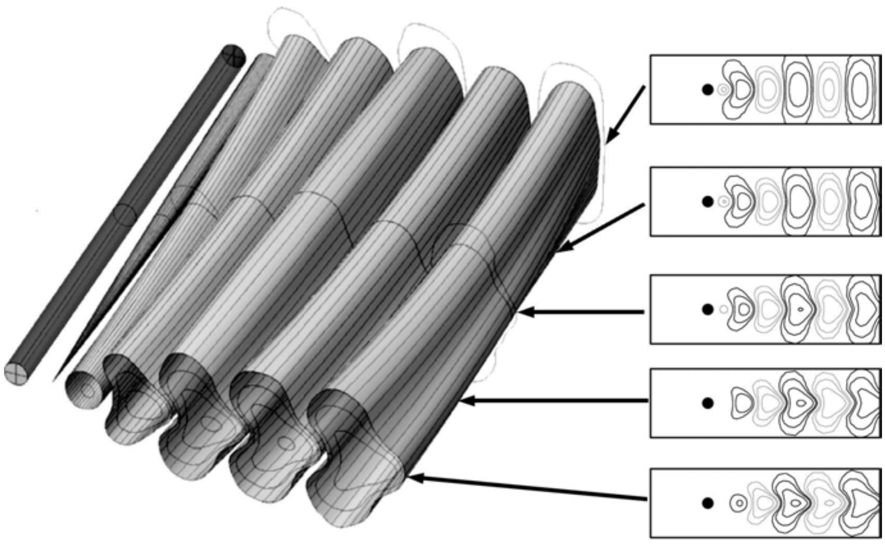

Figure 3: Principal sketch of continuous mode interpolation. Left: Transition between stabilty eigenmodes and POD modes. Right: streamlines of the intermediate states

In this technique, two eigenproblems A^0 and A^1, representing the terminal states, are linearly interpolated in $\kappa \in [0, 1]$ (7):

$$A^\kappa = A^0 + \kappa(A^1 - A^0), \tag{7}$$

In the case of POD modes, the matrices represent discretized Fredholm kernels (autocorrelation function) (8)

$$\mathbf{A}^\kappa(x, y) = \mathbf{u}_1^\kappa(x) \otimes \mathbf{u}_1^\kappa(y) + \mathbf{u}_2^\kappa(x) \otimes \mathbf{u}_2^\kappa(y) + ... \tag{8}$$

of Fredholm eigenproblem in space domain (9)

$$\int_\Omega \mathbf{A}(x, y)\mathbf{u}_i(y)dy = \lambda_i \mathbf{u}_i(x) \tag{9}$$

In the case of eigenmode interpolation, the matrices representing linearized Navier-Stokes equations are utilized.

Eigenvectors of interpolated eigenproblem (interpolated modes) $\mathbf{u}^\kappa$ can be used to model all the intermediate states between $\kappa = 0$ and $\kappa = 1$. In addition, the extrapolation of modes outside the design conditions is possible.

Interpolated modes enable 'least-order' Galerkin models keeping the dimension from a single operating condition but resolving several states (10).

$$\dot{a}_i^\kappa = \frac{1}{Re} \sum_{j=0}^{N} l_{ij}^\kappa a_j^\kappa + \sum_{j=0}^{N} \sum_{k=0}^{N} q_{ijk}^\kappa a_j^\kappa a_k^\kappa, \tag{10}$$

where

$$\dot{\kappa} = F(\kappa, \mathbf{a}^\kappa, t) \tag{11}$$

These models are especially well suited for control design. The results showing transients of circular cylinder flow modeled with the use of continuous mode interpolation are depicted in Fig. 2.

5 Least-order Galerkin model of NACA-0012 airfoil flow

The technique presented in previous section is applied here for laminar flow around NACA-0012 airfoil (Fig. 4). More technical details of the approach can be found in [19].

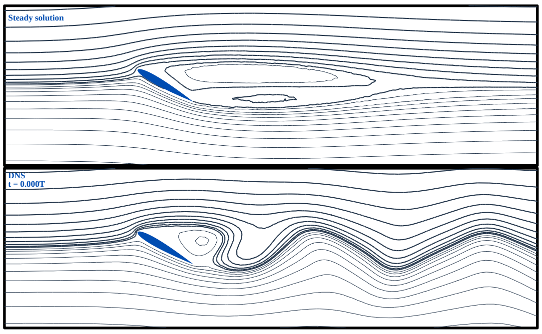

Figure 4: Flow around NACA-0012 airfoil. Top: streamlines of the steady solution. Bottom: streamlines of the snapshot from the periodic (limit-cycle) flow.

In present study, a number of different mode bases are considered in the construction of Reduced Order Models.

Empirical models use POD modes (Fig. 5) computed with the snapshot technique of Sirovich [20], from the snapshots of periodic flow. We analyze here models based on two and eight most-energetic POD.

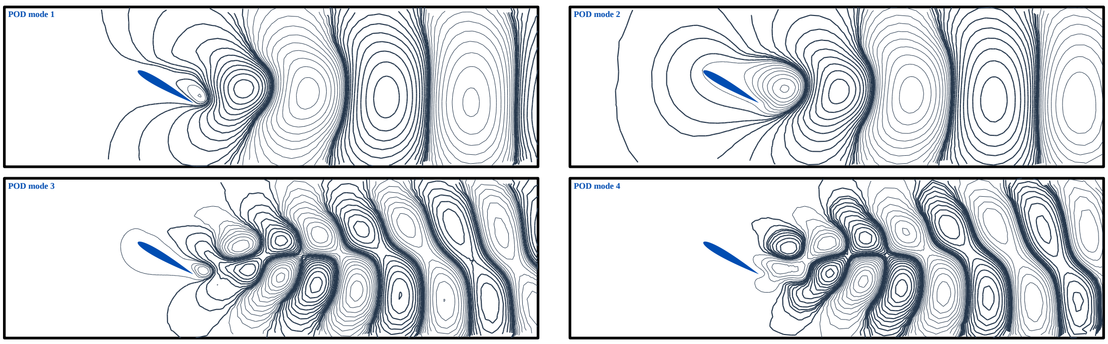

Figure 5: First 4 POD modes used in Galerkin modelling

The another flow model is designed with the stability eigenmodes, computed using steady and time-averaged solutions as a base flow (Fig. 6).

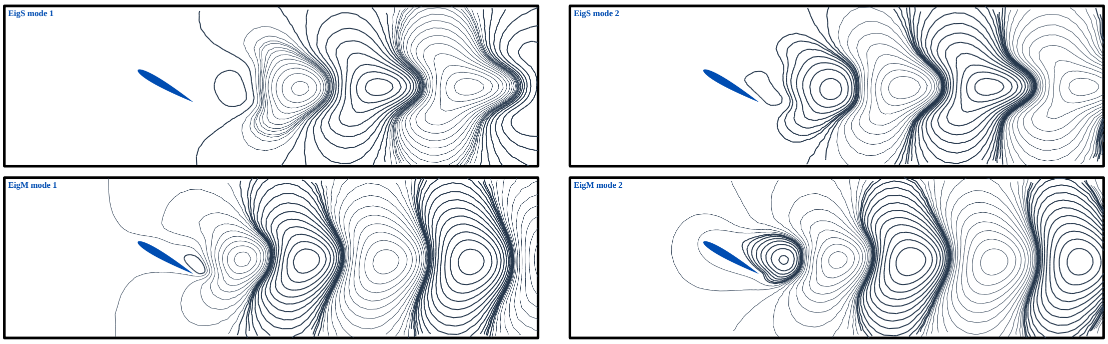

Figure 6: The most dominant eigenmodes based on steady solution (top, $\lambda = -0.147 \pm 0.720\imath$) and time-averaged solution (bottom, $\lambda = 0.018 \pm 0.915\imath$)

The last of the models presented in this paper is the least-dimensional model of two modes and continuous mode interpolation. The mean-field correction (shift mode) is employed for all presented here models to avoid the structural instability and the fragility. The comparison of all models is shown in figures 7 and 8.

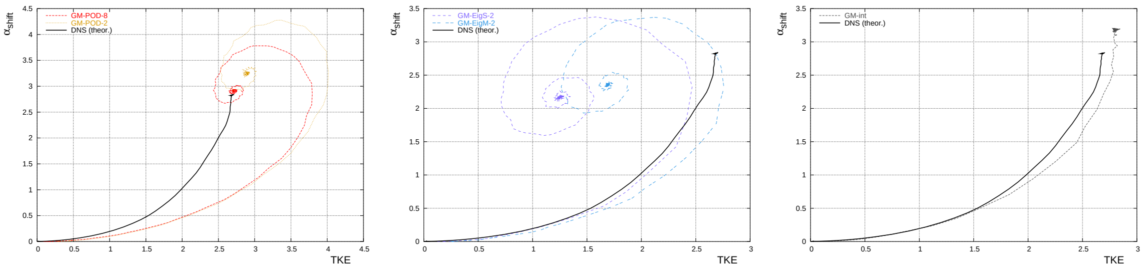

Figure 7: Shift-mode coefficient as a function of TKE for POD-based models (left), eigenmode-based models (middle), and interpolated model (right)

It can be seen in Fig. 8 that POD modes allow the reconstruction of Navier-Stokes attractor (limit cycle), but they are unable to reproduce the dynamical properties of transitional flow. For the flow states close to fixed point (steady solution, small values of shift-mode coefficient), the kinetic energy of the flow is overestimated - especially in POD-2 Galerkin model.

The models based on the two most unstable eigenmodes (Fig. 7, middle) reconstruct the flow states close to fixed-point (steady solution) and the transition to limit cycle better than POD Galerkin models. On the other hand the limit-cycle disturbance kinetic energy and shift-mode coefficients of periodic flow are significantly underestimated with these mode bases.

To take the adventage of both mode bases, interpolated model of the flow around NACA-0012 airfoil is used.

In the case of transition between unstable (steady solution) and stable (limit cycle oscillations) attractor, interpolation parameter κ is related to shift-mode amplitude (coefficient). $\kappa = 0$ represents steady state and dynamics described by the most stable stability eigenmodes, while $\kappa = 1$ is related to limit-cycle dynamics and POD modes.

Such a model preserves low dimensionality (4 equations for shift-mode amplitude, interpolated mode amplitudes and κ) and provides high accuracy in a wide range of operating conditions.

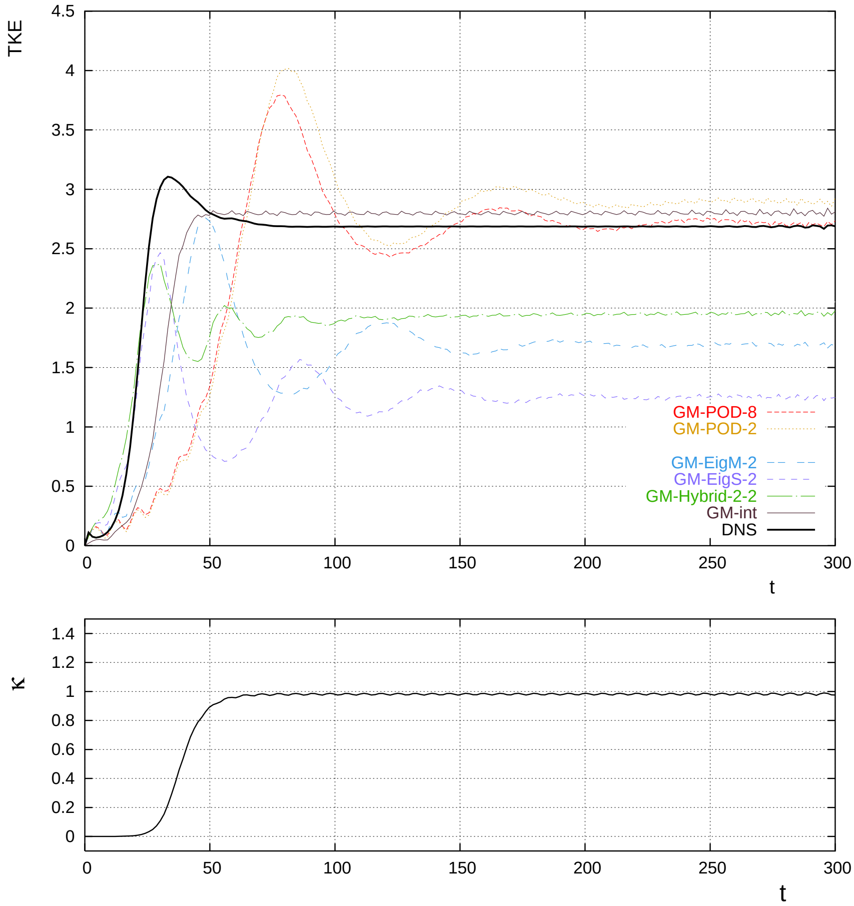

Figure 8: The variation of disturbance kinetic energy (TKE) for different Galerkin models and interpolation parameter κ in function of time, for transition from steady state to the limit cycle oscillation

6 Conclusions

In the present paper we emphasised the necessity of Reduced Order Models of flows in closed loop flow control as well as in the design process. ROM is able to deliver technically relevant answers in fraction of time necessary for full scale computations. Robust ROM of the flow is also the most important element of the online-capable feedback flow control. We concentrate here on assuring adequate dynamical properties of the model. Traditional methods of ROM construction, based on assemble of POD modes, are dynamically fragile and over-optimized at single operating conditions. Enrichment of the mode basis is one of the key technique of improvement. Shift mode, providing the missing direction from the fix point to the limit cycle assures the convergence of the

Reduced Order Galerkin Model. Hybrid models, employing stability eigenmodes further improve the dynamics. The construction of the least-dimensional flow model, preserving accuracy in a wide range of operating conditions, is possible with continues mode interpolation between POD and stability eigenmodes. A key enabler in ROMs presented here and flow control is the global stability analysis.

References

[1] A. Deane, I. Kevrekidis, G. Karniadakis, S. Orszag: Low-dimensional models for complex geometry flows: Application to grooved channels and circular cylinders. Phys. Fluids A **3** (1991) 2337–2354

[2] M. Morzyński, W. Stankiewicz, B.R. Noack, F. Thiele, and G. Tadmor: Generalized mean-field model with continuous mode interpolation for flow control. In *3rd AIAA Flow Control Conference*, San Francisco, Ca, USA, 5-8 June 2006, 2006. Invited AIAA-Paper 2006-3488.

[3] C. Rossow and N. Kroll, Opportunities for Next Generation Product Development, AIAA Aerospace Meeting, Reno NV, Jan 7-10, 2008, AIAA Paper 2008-0712.

[4] M. Bergmann, L. Cordier, J.P. Brancher: Optimal rotary control of the cylinder wake using proper orthogonal decomposition reduced order model. Phys. Fluids **17** (2005) 097101–097121

[5] B. Jørgensen, J. Sørensen, M. Brøns: Low-dimensional modeling of a driven cavity flow with two free parameters. Theoret. Comput. Fluid Dynamics **16** (2003) 299–317

[6] A. Khibnik, S. Narayanan, C. Jacobson, K. Lust: Analysis of low dimensional dynamics of flow separation. In Continuation Methods in Fluid Dynamics. Notes on Numerical Fluid Mechanics, vol. 74, Vieweg (2000) 167–178 Proceedings of ERCOFTAC and Euromech Colloquium 383, Aussois, France (1998)

[7] X. Ma, G. Karniadakis: A low-dimensional model for simulating three-dimensional cylinder flow. J. Fluid Mech. **458** (2002) 181–190

[8] S. Siegel, K. Cohen, J. Seigel, T. McLaughlin: Proper orthogonal decomposition snapshot selection for state estimation of feedback controlled flows. AIAA-Paper 2006-1400, AIAA Aerospace Sciences Meeting and Exhibit, Reno, Nevada, USA (2006)

[9] P. Holmes, J.L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge University Press, Cambridge, 1998.

[10] B.R. Noack, P. Papas, P.A. Monkewitz. The need for a pressure-term representation in empirical Galerkin models of incompressible shear-flows. *J. Fluid Mech.*, 523:339–365, 2005.

[11] B. R. Noack, M. Schlegel, B. Ahlborn, G. Mutschke, M. Morzyński, P. Comte, and G. Tadmor. A finite-time thermodynamics of unsteady fluid flows. *J. Non-Equilibr. Thermodyn.*, 33(2):103–148, 2008.

[12] N. Abdessemed, S. Sherwin, and V. Theofilis: Linear Stability of the flow past a low pressure turbine blade, *36 th AIAA Fluid Dynamics Conference and Exhibit*, 2006.

[13] J. Crouch, A. Garbaruk, and D. Magidov: Predicting the onset of flow unsteadiness based on global instability, *Journal of Computational Physics*, Vol. 224, No. 2, 2007, pp. 924–940.

[14] V. Theofilis: Advances in global linear instability analysis of nonparallel and three-dimensional flows, *Progress in Aerospace Sciences*, Vol. 39, No. 4, 2003, pp. 249–315.

[15] F. Giannetti and P. Luchini: Structural sensitivity of the first instability of the cylinder wake, *Journal of Fluid Mechanics*, Vol. 581, 2007, pp. 167–197.

[16] M. Morzyński and F. Thiele: Finite Element Method for Global Stability Analysis of 3D Flows. AIAA-Paper 2008-3865, 38th AIAA Fluid Dynamics Conference and Exhibition, Seattle, Washington, USA, 2008.

[17] B.R. Noack, K. Afanasiev, M. Morzyński, G. Tadmor, and F. Thiele: A hierarchy of low-dimensional models for the transient and post-transient cylinder wake. *J. Fluid Mech.*, 497:335–363, 2003.

[18] M. Morzyński, K. Afanasiev, and F. Thiele: Solution of the eigenvalue problems resulting from global non-parallel flow stability analysis. *Comput. Meth. Appl. Mech. Enrgrg.*, 169:161–176, 1999.

[19] W. Stankiewicz, M. Morzyński, B.R. Noack, G. Tadmor: Reduced Order Galerkin Models of Flow Around NACA-0012 Airfoil. *Mathem. Modelling and Analysis*, 13(1): 113–122, 2008.

[20] L. Sirovich: Turbulence and the dynamics of coherent structures. *Quart. Appl. Math.*, 45: 561–590, 1987.