ARTIFICIAL NUMERICAL DAMPING FOR LINEARIZED EULER EQUATION IMPLEMENTED IN RESIDUAL DISTRIBUTION METHOD

L. Koloszár*, N. Villedieu† T. Quintino‡

P. Rambaud§ and J. Anthoine¶

von Karman Institute for Fluid Dynamics, 1640 Rhode-St-Genèse, Belgium

Abstract

In this work we study the feasibility of Residual Distribution Method (RDM) to compute linear wave solutions. The Linearized Euler Equations are used therefore and it was found that weak numerical instability waves are formed ahead of the propagating waves. We present a Fourier analysis of acoustic modes for some Residual Distribution (RD) schemes for coupled space-time discretization. The numerical dispersion error and dissipation are calculated in order to identify the source of numerical instabilities. It is found that the multidimensional upwind schemes reduces the cross dissipation of the schemes (compared to the dimension splitting upwind schemes) resulting in less overal dissipation in case of two or three dimensional problems.Then, based on the Fourier analysis, we suggest an additional selective filtering (numerical dumping) which we support by numerical experiments.

1 Introduction

In case of higher order discretization, the wavelengths of the artificial waves which create instabilities are concentrated in a narrow band in the unresolved wave region of the numerical scheme [2]. The dominant wavelength can be identified by Fourier analysis of the scheme. In general, the Fourier analysis of the considered schemes can provide the number of nodes needed to resolve accurately a given wavelength with a given gridspacing. The analysis moreover give detailed description of the dissipation and the dispersion behaviour along the propagation direction.

The oscillations created by the underresolved waves can be eliminated by adding artificial numerical viscosity, however this treatment inevitably reduces the abilities of the given numerical scheme. Since the wavenumber found to lie in the under-resolved region of the numerical scheme it seems to be more appropriate to apply numerical viscosity just in that region. This idea was introduced by Tam and Shen[9] and the present work extends their filtering formulation for Residual Distribution Method.

First, we describe the residual schemes and the ways of discretizing unsteady problems. Then, through the Fourier analysis we show the abilities of these schemes. Based on the Fourier analysis a selective filtering is suggested in the third section and justified by numerical examples.

2 Numerical discretization

Residual Distribution Method is a numerical discretization technique somewhere between Finite Volume and Finite Element Method. The idea was introduced in 1982 by P.L. Roe [8] for the solu-

*Member of Doctoral Programme, Environmental and Applied Fluid Dynamics Department; koloszar@vki.ac.be

†Member of Doctoral Programme, Aeronautics and Aerospace Department

‡Dr, Senior Research Engineer, Aeronautics and Aerospace Department

§Dr, Assistant Professor, Environmental and Applied Fluid Dynamics Department

¶Dr, Associate Professor, Environmental and Applied Fluid Dynamics Department

tion of conservation laws on unstructured meshes. It provides true multidimensional propagation of information resulting in schemes with a rather small cross-diffusion.

The applied numerical method is described through the unsteady scalar advection problem:

$$\frac{\partial u}{\partial t} + \nabla\vec{\lambda} \cdot u = 0 \quad \forall (x, y, t) \in \Omega_t = \Omega \times [0, t_f] \tag{1}$$

In the space-time framework, time is considered as a third dimension and Ω_t is discretised by a succession of prismatic elements as shown on Figure 1. For any given function u, its restriction on the prism is defined by :

$$u^h(x, y, t) = \sum_l H^l(t) \sum_{i \in T} \psi_i(x, y)\, u_i^l\ , \tag{2}$$

where u_i^l is the value of u^h at node i and time t_l : $u_i^l = u^h(x_i, y_i, t_l)$, H^l is the $1D$ linear basis function and $\psi_i(x, y)$ denotes the (mesh dependent) linear continuous Lagrangian basis function. In each space-time element u^n is considered as known and u^{n+1} is the unknown.

The first step of the discretization is the computation of the residual on each space-time prism:

$$\Phi = \int_{t_n}^{t_{n+1}} \int_T \left(\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{\lambda} \cdot u \right) d\Omega\, dt \tag{3}$$

After some algebraic it yields to:

$$\Phi = \frac{|T|}{3} \sum_{i \in T} (u_i^n - u_i^{n+1}) + \frac{\Delta t}{2} ((\Phi^{\text{adv},T})^n + (\Phi^{\text{adv},T})^{n+1})$$

The residual Φ^{adv} can be computed either with integrating directly on the element or with the help of Gauss theorem by computing the contour integral. The later is the so-called Contour Residual Distribution (CRD) developed by Ricchiuto [6]. This formulation automatically ensures conservation so in this work we only consider CRD schemes. The residual is then:

$$\Phi^{adv} = \oint_{\partial T} \overrightarrow{\lambda} \cdot u \cdot \hat{n}\, dl\, d\Omega.$$

This formulation ensures conservativity automatically and in this article we only consider CRD schemes.

The second step is the distribution of the residual to the nodes of the prism. To respect the physical meaning of time, we do not want to distribute to the nodes of the level n. The consistency of the scheme is ensured by a constaint on the time step called past-shield condition. Under this condition it is possible to distribute the residual Φ only to the nodes of the level $n + 1$:

$$\begin{cases} \Phi_i^n & = 0 \\ \Phi_i^{n+1} & = \beta_i \Phi^K \end{cases}$$

The Residual distributive scheme in pratice is defined through the distribution coefficient β_i. The multidimensional upwinding property is the major difference to other methods like Finite Difference (FD) or Finite Volume (FV). With this property, the scheme follows better the original physical problem, improving the accuracy without increasing the number of degrees of freedom involved in the scheme's stencil. (This brings very low cross diffusion as was justified by the authors in a previous paper [5].)

In order to mimic the behaviour of exact solutions to scalar advection equation, multidimensional upwind schemes only distribute the residual to nodes which belong downstream with respect to the orientation of $\vec{\lambda}$. WeFor each time layer, we can define for every node an upwind parameter as:

$$k_i = \frac{1}{2}\overrightarrow{\lambda} \cdot \overrightarrow{n_i}; \tag{4}$$

where $\vec{n}_i$ is the inward normal to the edge of T facing node $i \in T$ as shown on Figure 2. The norm of $\vec{n}_i$ is equal to the length of the edge. With the help of the upwind parameters it is possible to construct special multi-dimensional upwinding schemes.

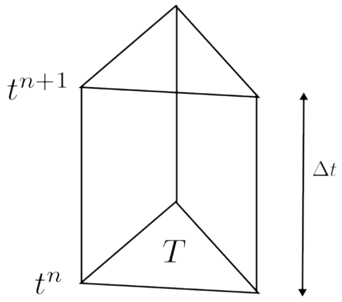


Figure 1: Space-time prism.

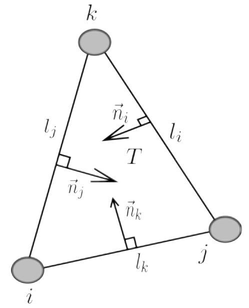


Figure 2: Definition of $\vec{n}_i$.

The same idea leads to $\tilde{k}_j$, the space-time upwind parameter for level n and $n+1$:

$$\begin{aligned} \tilde{k}_i^n &= \frac{\Delta t}{2}k_i - \frac{|T|}{3} \\ \tilde{k}_i^{n+1} &= \frac{\Delta t}{2}k_i + \frac{|T|}{3} \end{aligned} \tag{5}$$

where $\Delta t = t^{n+1} - t^n$, $|T|$ is the area of the element T and k_i is the upwind-parameter of the steady state.

The scheme used through the paper is called low diffusion A (LDA) because it is one of the less dissipative schemes constructed till now for RDM. Here we present the extension of LDA scheme to the space-time framework. The distribution coefficient of this scheme, is defined by:

$$\beta_i^{\mathrm{LDA}} = \frac{\tilde{k}_i^{n+1,+}}{\sum_{j\in T} \tilde{k}_j^{n+1,+}}$$

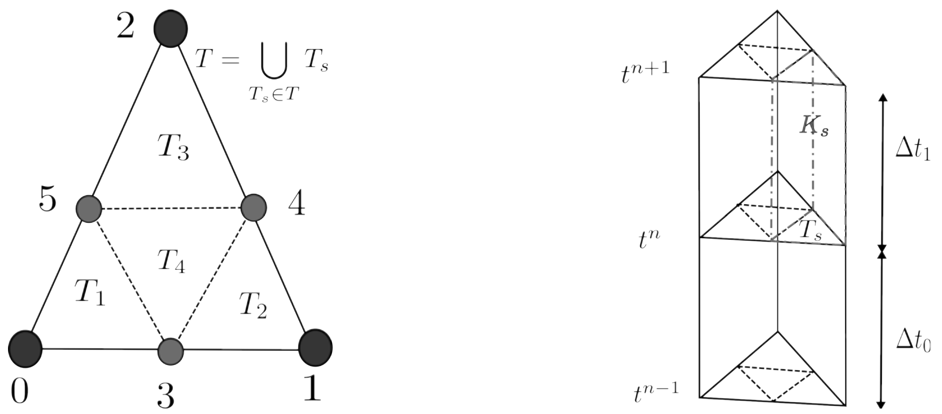
The LDA scheme is a multidimensional upwind scheme and in case of linear elements it is found to be 2^{nd} order accurate, while in case of quadratic elements its order is around 3. After distribution of the residual, we assemble all the contributions to each node into the nodal Equation 6. This system of equations is solved by pseudo-time iterations as in Equation 7.

$$\sum_{T,i\in T} \Phi_i^{n+1} = 0 \tag{6}$$

$$u_i^{n+1,\kappa+1} = U_i^{n+1,\kappa} + \frac{\Delta\tau}{C_i}\left(\sum_{T,i\in T} \Phi_i^{T,n+1}\right)^{\kappa} \tag{7}$$

2.1 Extension to high order discretization

We use the approach of [7] to extend space-time schemes to high order discretization. To provide high order of accuracy both in space and in time we combine quadratic triangular elements in space with a quadratic discretization of time. This means that each triangular element in space is equipped with 6 degrees of freedom (see figure 3(a)). We split this triangle in four sub-elements $\{T_s\}_{s=1,4}$ as on Figure 3(a). This yields to the new space-time prism of figure 3(b) that has



(a) Quadratic elements in space

(b) Quadratic space-time element

Figure 3: High order elements

three levels in time. Each level is composed of quadratic triangles in space. In this prism u is approximated in the same way as before, by Equation 2, where u_i^l is the value of u^h at node i and time t_l : $u_i^l = u^h(x_i, y_i, t_l)$ like before, but $\psi_i(x, y)$ now denotes the (mesh dependent) quadratic continuous Lagrangian basis function, and H^l is the $1D$ quadratic basis function. We decompose this prism in sub-prisms which levels are the sub-element of the *quadratic* triangle (as illustrated on Figure 3(b)). In each of this sub-prism we can still define a space-time upwind parameters in the same way as for the linear elements (Equation 1) but now the upwind parameter k_i is defined with the well scaled normals of the sub-triangle T_s. At each time iteration u^{n-1} and u^n are known and we want to compute u^{n+1} using the usual steps. First, we compute the residual on each sub-prism between t^{n+1} and t^n:

$$\Phi^{K_s} = \int_{t_n}^{t_{n+1}} \int_{T_s} \left(\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{\lambda} \cdot u \right) d\Omega \, dt$$

Then, the residual is distributed to all the nodes of the sub-triangle T_s of the level $n + 1$. Finally, we use pseudo-time iterations to solve the final system as before.

3 Wavenumber analysis of the LDA scheme

The Fourier-analysis of the discretization method can provide a detailed information about the numerical response of the considered scheme to periodic wave excitation. Usually, aerodynamically generated noise is a decomposition of different sound wave modes. Thus, any numerical method used to compute noise generation and/or propagation must be able to represent these modes. Therefore it is necessary to perform a wavelength based numerical analysis in order to identify the limits of the chosen discretization. The perturbation involved in acoustic waves are very small so the response of the acoustic field is linear [1]. A main consequence is that there is no interaction between different acoustic waves, so it is enough to see the numerical response of the system to one mode. The dispersion and dissipation of the space operator detached from the time discretization can be found in Koloszar *et al.*[5]. In this paper the Fourier analysis of the full discretization is provided to get an insight to the behaviour of the complete system in case of linear and quadratic elements. Consider the two-dimensional unsteady scalar advection equation (Equation 1). Suppose that the solution is periodic over the domain $[0, L]^2$ and the grid spacing is h for both directions on a structured triangulation. Furthermore, suppose that the variable is periodic in time as well and the domain is extended in the time dimension - defined through the space-time approach - with grid spacing $\Delta t = CFL \cdot h / |\overrightarrow{\lambda}|$. The periodic variable u can be expressed as:

$$u(t,x,y) = \sum_{n=-\frac{L}{2}}^{\frac{L}{2}} \hat{u}^n e^{2\pi\imath\left(\frac{k_x^n x}{L} + \frac{k_y^n y}{L} + \frac{k_t^n t}{T}\right)}$$

Equation 6 is a discrete relationship between the nodes, but it is not the discrete form of the original partial differential equation (Equation 1). The integration over the element is included as well resulting a discrete formulation to Equation 3. To be able to reconstruct the discrete formulation of the original partial differencial equation, one has to follow three steps:

- Recast the formula to a Finite Different (quadrilateral) grid:

$$\int_{t_n-1}^{t_n} \int_{T_s} (\overrightarrow{\lambda} \nabla u)\, d\Omega\, dt = \frac{\Delta x}{2} \sum_{n,i,j \in T_s} (\tau_{n,i,j} + a \cdot \alpha_{n,i,j} + b \cdot \beta_{n,i,j}) \cdot u_{n,i,j}$$

- The grid has equidistant spacing so the right hand side do not depend on the global x and y coordinates, rather on the relative distance between the nodes set by the nodes involved in the given stencil, so take the derivative in x and y of both side gives the derivative along the propagation direction:

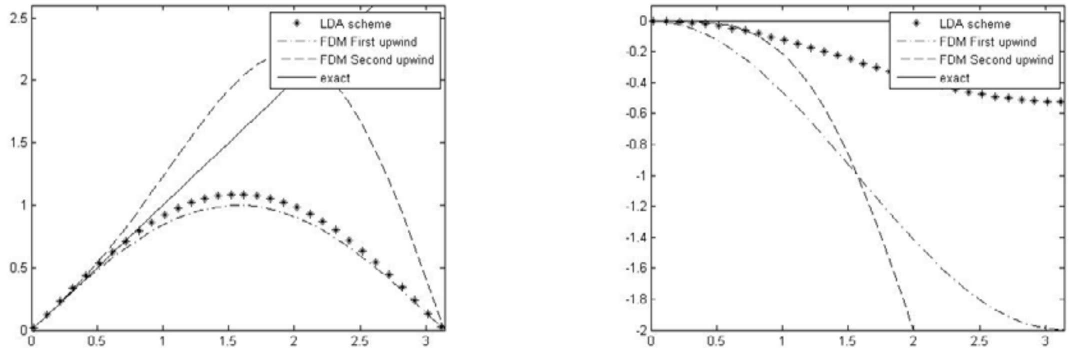
$$\overrightarrow{\lambda} \nabla u = \frac{1}{2\Delta x} \sum_{n,i,j \in T_s} (\tau_{n,i,j} + a \cdot \alpha_{n,i,j} + b \cdot \beta_{n,i,j}) \cdot u_{n,i,j} \tag{8}$$

- Take the 3D Fourier transform of both side of Equation 8 and after reordering one can obtain a relationship between the physical and the numerical wavenumber:

$$\tau + a \cdot \alpha + b \cdot \beta = -\imath \frac{\sum_{n,i,j \in T_s} (\tau_{n,i,j} + a \cdot \alpha_{n,i,j} + b \cdot \beta_{n,i,j}) \cdot e^{(n\tau\Delta t + i\alpha\Delta x + j\beta\Delta y)}}{2\Delta x}$$

Figure 4 shows the relations between the exact (x-axis) and the numerical (y axis) wavenumbers over the interval 0 to π, both for dispersion and dissipation. The linear LDA scheme (time and space discretization, CFL=0.7) is compared with the first and second order FD formulations (only space discretization). Due to multidimensional upwinding the dissipation is less along the streamlines than in case of dimensional splitting techniques. Although less significant there is also small improvement in the dispersion relation as well. For up to the wavenumber 0.75 the numerical wavenumber is very close to the exact one, thus the linear LDA scheme can give an adequate approximation to the original partial differential equation for waves with wavelength longer than 8 mesh spacings.

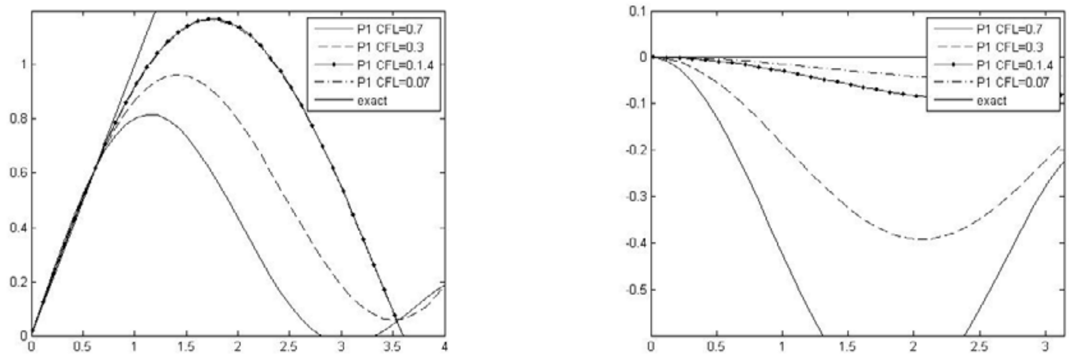
Figure 5 compares the LDA scheme over linear and quadratic triangular elements. Now the plots are given along the space-time streamlines, so along the line defined by the space-time



(a) Dispersion relation (b) Dissipation

Figure 4: Relations between the exact (x-axis) and the numerical wavenumbers (y axis) for linear LDA, FD 1st order and second order upwind schemes.

advection vector $\overrightarrow{\lambda} = [1\ a\ b]$. In this way the difference between the behaviour of LDA scheme over the linear and quadratic elements are more visible. In the legend one can see that there is an order difference between the CFL number for linear and quadratic elements. This is due to the past-shield condition. It is possible to overcome on this limitation by double time-stepping [4]. Figure 5 shows that in case of the linear elements the dispersion is more sensible to the CFL number, while in case of the quadratic element the dispersion is almost the same however the dissipation reduces significantly with the CFL number.



(a) Dispersion relation (b) Dissipation

Figure 5: Relations between the exact (x-axis) and the numerical wavenumbers (y axis) for linear and quadratic LDA scheme.

4 Artificial numerical damping

The formulation is based on the work of Tam and Shen [9] and extended for Residual Distribution Method. In the original Finite Difference formulation a smoothing operator is introduced as source term:

$$Source_i^{FDM} = -\frac{\nu_a}{\Delta x^2} \sum_{j=-3}^{3} c_j u_{i+j}$$

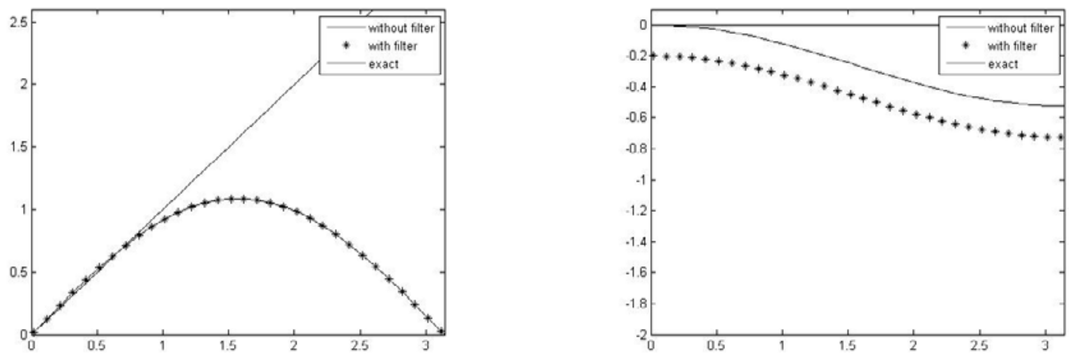
So a selective artificial damping is defined as artificial dissipation in a fixed wavenumber region. In order to achive this nice feature, the optimization of the filter is done in the frequency space: the coefficient c_j is adjusted to activate the filtering only in the desired wavenumber region.

The filtering operator is introduced as a source term in Residual Distribution Method as well, however due to the non-uniform mesh the generalization of the filtering is not straightforward. The filter has to rely on relative internode distances instead of grid spacing between them, like in the Finite Different formulation. Two different types of filtering are discussed here: the Laplacian filtering and the artificial selective damping.

In case of the Laplacian filter the following term is discretized through Residual Distribution Method:

$$Source_i^L = -\frac{\nu_a}{\Delta x^2} \nabla \overrightarrow{\lambda} \nabla u$$

Through the Fourier analysis of the Laplacian filter two major observations were made. First, the Laplacian filtering (Figure 6) does not affect the dispersion relation of the given scheme (now the LDA scheme over linear elements) it just adds dissipation gradually for all the wavenumbers. Definitly, this kind of filter won't change the propagation speed of the acoustic waves, however it will dissipate all the wavelengthes intensively, not just the high-frequency oscillations.



(a) Dispersion relation

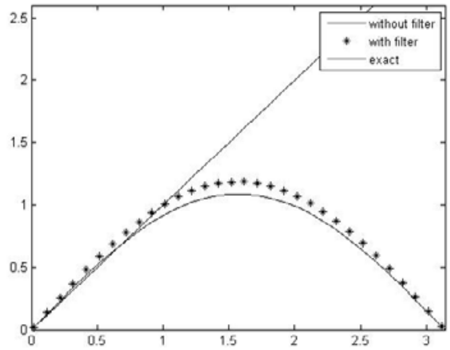
(b) Dissipation

Figure 6: Relations between the exact (x-axis) and the numerical wavenumbers (y axis) for Laplacian filtering.

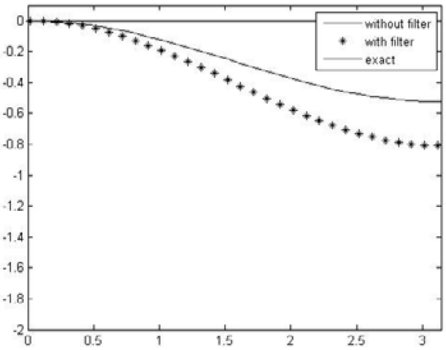
To weight the dissipation more in the high frequency region a form close to the original operator has been implemented:

$$Source_i^{SEL} = -\frac{\nu_a}{\Delta x^2} \cdot \sum_{T, i \in T} n_i^T \sum_{j \in T} c_j u_j$$

With $n_i = n_x + n_y + n_z$ belongs to node i and $c_j = |\overrightarrow{x_i} - \overrightarrow{x_j}|$, based on the distance between the target node $\overrightarrow{x_i}$ and the other nodes $\overrightarrow{x_j}$ involved in the discretization. The properties of the filter can be seen in Figure 7. In the left hand side the dispersion relation shows that the filter do change slightly the dispersion relation of the original scheme. However on the right figure one can see that the dissipation is unchanged for the long waves and gradually increased for the short ones. So this filter is more suitable for substructing high-frequency numerical noise without contaminate the solution too much.



(a) Dispersion relation



(b) Dissipation

Figure 7: Relations between the exact (x-axis) and the numerical wavenumbers (y axis) for artificial selective damping.

5 Computational results

The previously described method has been applied to discretize the Linearized Euler Equations. We consider these equations in two spatial directions, as derived by Bailly *et al.* [3] for inhomogeneous mean flow written in conservative variables:

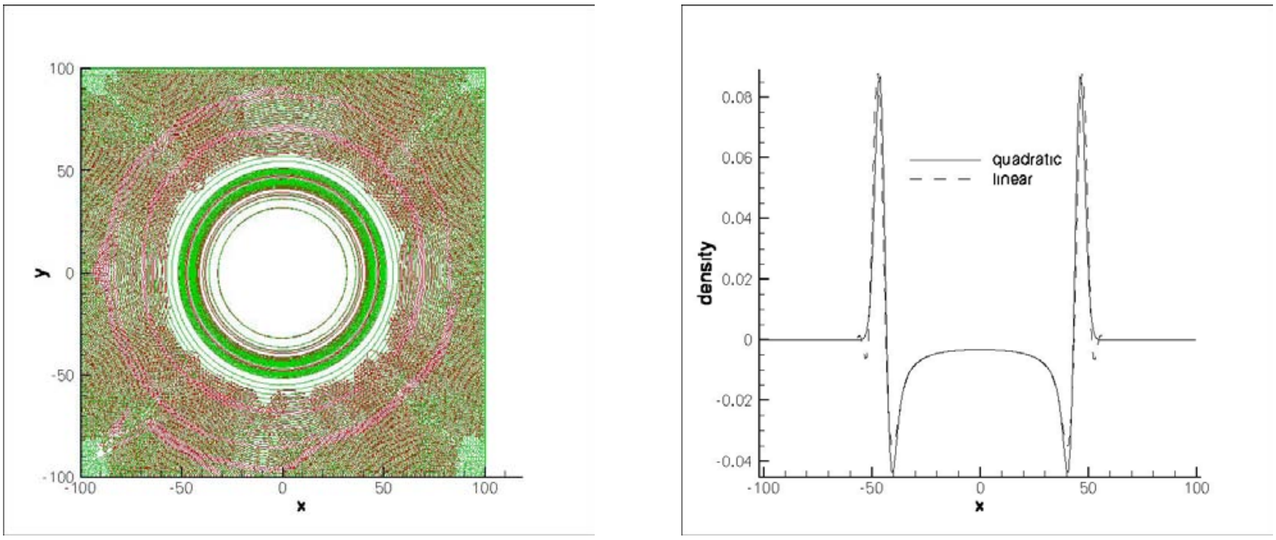
$$\frac{\partial \overrightarrow{U}}{\partial t} + \mathbf{A}\frac{\partial \overrightarrow{U}}{\partial x} + \mathbf{B}\frac{\partial \overrightarrow{U}}{\partial y} + \overrightarrow{H} = \overrightarrow{S} \tag{9}$$

The vectors and matrices read as follows:

$$\overrightarrow{U} = \begin{bmatrix} \rho \\ \rho_0 u \\ \rho_0 u \\ p \end{bmatrix}, \mathbf{A} = \frac{\partial \overrightarrow{F}}{\partial \overrightarrow{U}} = \begin{bmatrix} u_0 & 1 & 0 & 0 \\ 0 & u_0 & 0 & 1 \\ 0 & 0 & u_0 & 0 \\ 0 & c^2 & 0 & u_0 \end{bmatrix}, \mathbf{B} = \frac{\partial \overrightarrow{G}}{\partial \overrightarrow{U}} = \begin{bmatrix} v_0 & 0 & 1 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & 1 \\ 0 & 0 & c^2 & v_0 \end{bmatrix}.$$

$$\overrightarrow{H} = \begin{bmatrix} 0 \\ (\rho' u_0 + \rho_0 u')\frac{\partial u_0}{\partial x} + (\rho' v_0 + \rho_0 v')\frac{\partial u_0}{\partial x} \\ (\rho' u_0 + \rho_0 u')\frac{\partial v_0}{\partial x} + (\rho' v_0 + \rho_0 v')\frac{\partial v_0}{\partial x} \\ (\gamma - 1)p'(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial x}) - (\gamma - 1)(u'\frac{\partial p_0}{\partial x} + v'\frac{\partial p_0}{\partial x}) \end{bmatrix}.$$

And $\overrightarrow{S}$ represents the artificial sources such as monopole, dipole, etc. To preserve numerical stability the terms containing the derivatives of the background flow ($\overrightarrow{H}$) are treated as source terms. To illustrate the effect of the two elements (linear, quadratic), as well as the different filtering techniques of the numerical solution an acoustic pulse centered at the origin is computed. The following initial value problem is solved: $\rho = e^{-(ln2)\frac{x^2+y^2}{9}}$ $u = v = 0$ $p = e^{-(ln2)\frac{x^2+y^2}{9}}$ The pulse is placed into a quiscent flow. In order to fullfil the past-shield condition, the highest CFL number $CFL = 0.7$ was used in case of linear elements and $CFL = 0.07$ for the quadratic elements. The computational domain extends from $-100 \leq x, y \leq 100$ embedded in free space and discretized by 201X201 nodes. The perturbation velocities, density and pressure are normalized by c (speed of sound), ρ_0 (ambient density) and, $\rho_0 c^2$ respectively. To compare the linear and the quadratic implementations, the same time step with $CFL = 0.07$ has been used to compute the propagation of the acoustic pulse. The time instant of $t = 45$ is shown on Figure 8. On the contour plot the instability waves due to the discretization are clearly visible on the front of the pulse.



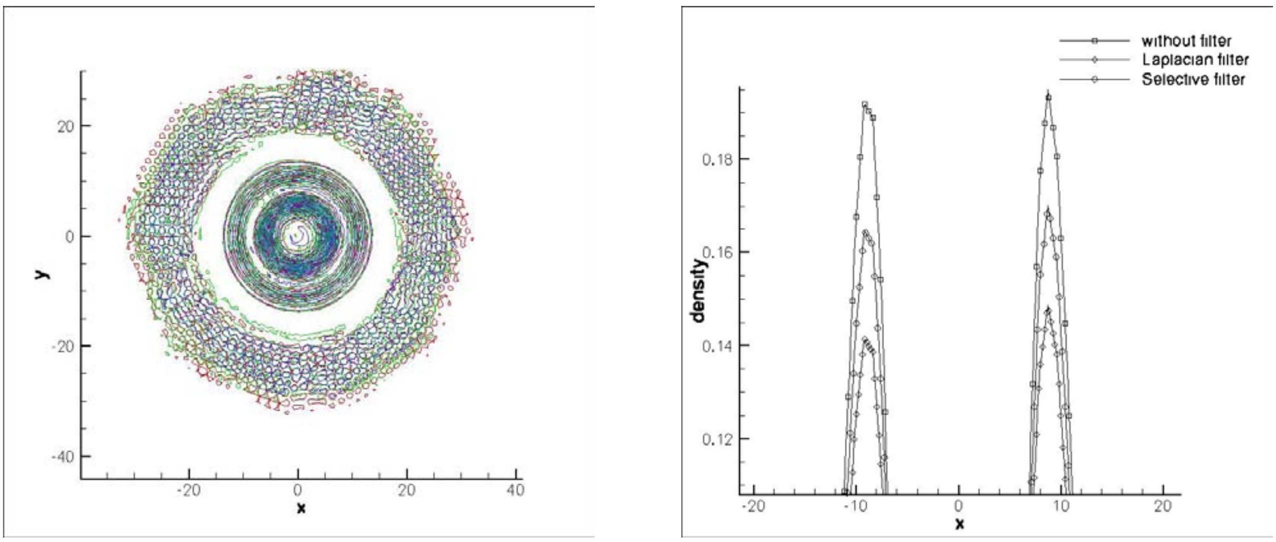
(a) Contours of pressure with contours between -0.18 and 0.18 plus $10^{-}8$.

(b) Plot along the x-axis.

Figure 8: Results with linear and quadratic elements.

Comparing the density signal along the x-axis (Figure 8 (b)) shows the advantage of the quadratic element over the linear one. The small CFL number due to the very thin prisms cause numerical instability problems in the case of the linear elements. This additional waves are not the "regular"instability waves due to the discretization and they are not present in case of higher CFL numbers, they are simply the side-effect of the ill-shaped elements.

In case of the filtering the main objective is to remove the high frequency numerical waves without modify much the original signal. As shown in Figure 9, the area where these waves are still active is reduced by both the Laplacian and the selective filter. However, if we plot the signals along the x-axis it can be observed that the selective filter destroys the original signal less than the Laplacian filter.



(a) Contours of pressure, red - without filter, green - Laplacian filtering, lue - selective damping.

(b) Plot along the x-axis.

Figure 9: Effect of filtering.

6 Conclusions

This paper shows that the Residual Distribution Method is an alternative method to resolve acoustic waves through Linearized Euler Equations. Due to the multidimensional upwind formulation, high accuracy is achieved with linear and quadratic discretizations. The space-time formulation ensures the consistent space and time discretization related to the original partial differential equations. Fourier analysis showed that multidimensional upwinding has less dissipation than dimensional splitting techniques with the same stencil size. Artificial selective damping has been adapted to the Residual Distribution Method, however due to the in built dissipation of upwind schemes the amplitude of these waves are so small (seven order smaller) that there is strong need for it on the testcases computed so far. The future work will focus to the developement of appropriate non-reflecting boundary treatment.

7 Acknowledgement

This research is supported through the Marie Curie Research Training Network Project AETHER (contract no MRTN-CT-2006-035713) and the SBO project CAPRICORN.

References

[1] *Sound and Sources of Sound.* Ellis Horwood Limited, 1983.

[2] Icase/larc workshop on benchmark problems in computational aeroacoustics (caa). *NASA CP 3300*, 1995.

[3] C. Bailly and D. Juvé. Numerical solution of acoustic propagation problems linearized euler's equations. *6th AIAA/CEAS Aeroacoustic Conference*, 2000.

[4] A. G. Csik. *Upwind Residual Distribution Schemes for General Hyperbolic Conservation Laws and Application to Ideal Magnetohydrodynamics.* PhD thesis, Katholike Universiteit Leuven, 2002.

[5] L. Koloszar, N. Villedieu, T. Quintino, J. Anthoine, and P. Rambaud. Application of residual distribution method for acoustic wave propagation. *AIAA 2009-3116*, 2009.

[6] M. Ricchiuto. *Construction and Analysis of Compact Residual Discretizations for Conservation Laws on Unstructured Meshes.* PhD thesis, Université Libre de Bruxelles, 2005.

[7] M. Ricchiuto, R. Abgrall, and H. Deconinck. Construction of very high order residual distributive schemes for unsteady scalar advection : preliminary results. In *33rd computational fluid dynamics-novel methods for solving convection dominated systems*, volume 5. von Karman Institute for Fluid Dynamics, 2003.

[8] P. L. Roe. Fluctuations and signals - a framework for numerical evolution problems. *Mumerical Methods for Fluid Dynamics*, 1982.

[9] C. K. W. Tam and H. Shen. Direct computation of nonlinear acoustic pulses using high-order finite difference schemes. *AIAA 93-4325*, 1993.