Mean stress effect from Drucker-Prager micro-plasticity criterion

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ABSTRACT

Snecma, in the frame of its developments and production activities related to rocket engines for the Ariane 5 european space launcher has to deal with high cycles fatigue phenomena at high stress ratio. Indeed, several parts of these engines, especially in the cryogenic turbopumps are submitted to high mean multiaxial stresses (pressure, centrifugation), superimposed to high frequency dynamic stresses with constant or variable amplitude, random or not. The thermal environment is stabilized at high (<1000K) or very low temperature (20K). The mean stresses can eventually lead to localized plastification.

In this context, in cooperation with EDF and AREVA-NP which are also confronted to HCF phenomena for nuclear applications in anisothermal environments, improvements of the LMT Cachan two scale model are being conducted. This article aims at focusing the improvements related to the way the mean stress effect is taken into account.

On the idea that fatigue damage is localized at the microscopic scale, a scale smaller than the mesoscopic one of the Representative Volume Element (RVE), a three-dimensional incremental two scale damage model has been proposed for High Cycle Fatigue applications [1,2] and extended to anisothermal cases and to thermo-mechanical fatigue [3]. Mean stress effect was first introduced by the modelling of microdefects (or microcracks) closure by means of a microdefects closure parameter "*h*" in the expression of the energy density release rate. This parameter appeared to be not sufficient to represent realistic fatigue behaviour, especially for positive stress ratio (0.1 to 0.9). The use of the Drucker-Prager micro-plasticity criteria at the micro scale of the two scale model makes it now able to properly take into account the mean stress effect in calculations performed with the associated fatigue "DAMAGE" post-processor.

The Drucker-Prager micro-plasticity criteria presents the advantage to use the first stress invariant "$Tr\tilde{S}^{\mu}$" multiplied by a coefficient "k" to identify (Figure 2).
Haigh and Goodman diagrams are gained by the time integration of the constitutive equations over the whole loading thanks to this post-processor.

KEYWORDS

Damage; mean stress effect; two scale model; fatigue; post-processor.

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TWO SCALE DAMAGE MODEL

On the remark that High Cycle Fatigue, either thermally or mechanically activated, occurs for an elastic regime at the RVE scale, the mesoscale of continuum mechanics, a two scale damage model has been built [1,2]. It accounts for micro-plasticity and micro-damage at the defects scale or microscale. The model is phenomenological, describing micro-plasticity with classical 3D von Mises plasticity equations, describing micro-damage by Lemaitre damage evolution law $\dot{D} = \left(Y/S\right)^s \dot{p}$ of damage governed by the accumulated plastic strain rate $\dot{p}$, with Y the elastic energy density rate and S and s the damage parameters. A scale transition law makes the link between both mesoscopic and microscopic scales (Fig.1).

The general principles for building a two scale damage model for thermal and/or thermo-mechanical fatigue are as follows,

- at the mesoscale, the scale of the RVE of continuum mechanics, the behavior is considered as thermo-elastic, the material engineering yield stress σ_y being usually not reached in High Cycle Fatigue (accordingly called elastic fatigue),
- the microscale is the defects scale, defects conceptually gathered as a weak inclusion imbedded in previous RVE. The behavior at microscale is thermo-elasto-plasticity coupled with damage, the weakness of the inclusion being represented by a yield stress at microscale σ_y^μ taken equal to the asymptotic fatigue limit of the material σ_f^∞.

At the mesoscale, the stresses are denoted $\boldsymbol{\sigma}$, the total, elastic and plastic strains are $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^e$, and $\boldsymbol{\varepsilon}^p$. They are known from a thermo-elastic Finite Element (FE) computation as for High Cycle Fatigue one has most often $\boldsymbol{\varepsilon}^p \approx 0$. An initial plastification can be handled by the two scale damage model by considering constant non zero mesoscopic plastic strains $\boldsymbol{\varepsilon}^p$, for instance gained from the nonlinear FE analysis of the initial yielding.

The values at the microscale have an upper-script "μ". For High Cycle Fatigue, with plasticity and damage assumed to occur at the microscale only, one has $\boldsymbol{\varepsilon}^{p\mu} \neq 0$, $0 < D < 1$, where for simplicity the damage variable at the microscale has no upper-script ($D=D^\mu$).

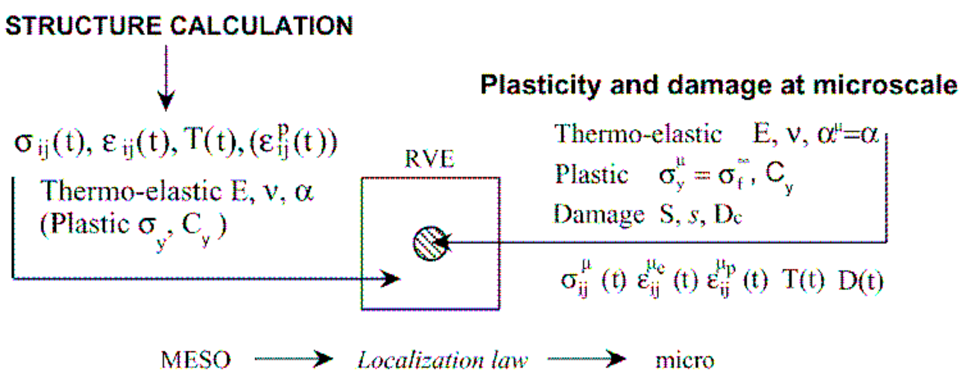

Figure 1 - Micro element imbedded in a thermo-elastic Representative Volume Element.

Thermo-elastic behavior at mesoscale

The thermoelastic law for the RVE reads :

$$\epsilon^e = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\mathrm{tr}\,\sigma\,\mathbf{1} + \alpha(T - T_{\mathrm{ref}})\mathbf{1}$$

with E the Young modulus, ν the Poisson ratio, α the thermal expansion coefficient and T_{ref} the reference temperature. The non homogeneous temperature and eventually time dependent field in a structure T(x,t) is usually determined from an initial heat transfer computation. The mechanical properties E, ν, α may depend on the temperature. The shear and bulk modulus of the material will respectively be $G=E/2(1+\nu)$ and $K=E/3(1-2\nu)$.

Plasticity and damage at microscale

A law of thermo-elasto-plasticity coupled with damage is considered at microscale. No viscosity is considered as for the applications in mind the temperature will remain much lower than one third of the melting temperature. The elasticity law reads then (recall that μ-upper-script stands for "variable at microscale") :

$$\epsilon^{\mu e} = \frac{1+\nu}{E}\frac{\sigma^{\mu}}{1-D} - \frac{\nu}{E}\frac{\operatorname{tr}\sigma^{\mu}}{1-D}\mathbf{1} + \alpha^{\mu}(T - T_{\text{ref}})\mathbf{1}$$

where the thermal expansion coefficient α^{μ} is taken next equal to the meso coefficient α. In the yield criterion, the hardening $\mathbf{X}^{\mu}$ is kinematic, linear, and the yield stress is the asymptotic fatigue limit of the material, denoted σ_f^{∞}

$$f^{\mu} = (\tilde{\boldsymbol{\sigma}}^{\mu} - \boldsymbol{X}^{\mu})_{\text{eq}} - \sigma_f^{\infty}$$

with $(.)_{eq}$ von Mises norm and where the elasticity domain is defined by $f^{\mu}<0$.

The set of constitutive equations at microscale is then :

$$\begin{cases} \boldsymbol{\epsilon}^{\mu} = \boldsymbol{\epsilon}^{\mu e} + \boldsymbol{\epsilon}^{\mu p}, \\ \boldsymbol{\epsilon}^{\mu e} = \dfrac{1+\nu}{E}\tilde{\boldsymbol{\sigma}}^{\mu} - \dfrac{\nu}{E}\operatorname{tr}\tilde{\boldsymbol{\sigma}}^{\mu}\mathbf{1} + \alpha(T - T_{\text{ref}})\mathbf{1}, \\ \dot{\boldsymbol{\epsilon}}^{\mu p} = \dfrac{3}{2}\dfrac{\tilde{\boldsymbol{\sigma}}^{\mu D} - \boldsymbol{X}^{\mu}}{(\tilde{\boldsymbol{\sigma}}^{\mu} - \boldsymbol{X}^{\mu})_{\text{eq}}}\dot{p}^{\mu}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\boldsymbol{X}^{\mu}}{C_y}\right) = \dfrac{2}{3}\dot{\boldsymbol{\epsilon}}^{\mu p}(1-D), \\ \dot{D} = \left(\dfrac{Y^{\mu}}{S}\right)^{s}\dot{p}^{\mu} \quad \text{if } p^{\mu} > p_D^{\mu} \text{ or if } w_s^{\mu} > w_D, \\ D = D_c \rightarrow \text{crack initiation} \end{cases}$$

With the plastic modulus C_y, the damage strength S, the damage exponent s as temperature dependent material parameters.
In previous laws, p^{μ} is the accumulated plastic strain at micro-scale, w_s^{μ} is the stored energy density and p_D^{μ} and w_D are the corresponding damage thresholds. A damage threshold in terms of accumulated plastic strain is loading dependent so that a threshold in terms of stored energy can advantageously be considered. A crack is initiated when D reaches the critical damage D_c.

Localization law coupled with damage and temperature

The scale transition meso → micro is governed by modified Eshelby-Kröner localization law [3,4,5]:

$$\boldsymbol{\epsilon}^{\mu D} = \frac{1}{1-bD}\left[\boldsymbol{\epsilon}^{D} + b\big((1-D)\boldsymbol{\epsilon}^{\mu p} - \boldsymbol{\epsilon}^{p}\big)\right],$$

$$\epsilon_H^{\mu} = \frac{1}{1-aD}\left[\epsilon_H + a\big((1-D)\alpha^{\mu} - \alpha\big)(T - T_{\text{ref}})\right]$$

where $(.)^D=(.)-(1/3)\operatorname{tr}(.)\mathbf{1}$ stands for the deviatoric part of a tensor, $(.)_H=(1/3)\operatorname{tr}(.)$ for the hydrostatic part. By considering $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^D + \varepsilon_H\mathbf{1}$, $\boldsymbol{\varepsilon}^{\mu D} + \varepsilon^{\mu}_H\,\mathbf{1}$ the localization law reads:

$$\boldsymbol{\epsilon}^{\mu} = \frac{1}{1-bD}\left[\boldsymbol{\epsilon} + \frac{(a-b)D}{3(1-aD)}\operatorname{tr}\boldsymbol{\epsilon}\mathbf{1} + b\big((1-D)\boldsymbol{\epsilon}^{\mu p} - \boldsymbol{\epsilon}^{p}\big)\right] + \frac{a((1-D)\alpha^{\mu} - \alpha)}{1-aD}(T - T_{\text{ref}})\mathbf{1}$$

with *a* et *b* the Eshelby parameters for a spherical inclusion,

$$a = \frac{1+\nu}{3(1-\nu)}, \qquad b = \frac{2}{15}\frac{4-5\nu}{1-\nu}$$

MEAN STRESS EFFECT IN THE INCREMENTAL TWO SCALE DAMAGE MODEL

Uniaxial mean stress is defined as $\sigma_m=(\sigma_{min}+\sigma_{max})/2$ and is of course related to the stress ratio $R_\sigma=\sigma_{min}/\sigma_{max}$. It is observed that for negative mean stress, damage evolution is smaller than for positive mean stress. It is important to take this effect into account especially for high mean stress loadings. Two main causes responsible for this effect are presented hereafter.

Micro defects closure parameter *h*

So far, a difference between positive or negative mean stress loadings was introduced by the modelling of microdefects (or microcracks) closure by means of a microdefects closure parameter *"h"* in the expression of the energy density release rate:

$$Y^\mu = \frac{1+\nu}{2E}\left[\frac{\langle\sigma^\mu\rangle^+ : \langle\sigma^\mu\rangle^+}{(1-D)^2} + h\frac{\langle\sigma^\mu\rangle^- : \langle\sigma^\mu\rangle^-}{(1-hD)^2}\right] - \frac{\nu}{2E}\left[\frac{\langle \mathrm{tr}\,\sigma^\mu\rangle^2}{(1-D)^2} + h\frac{\langle -\mathrm{tr}\,\sigma^\mu\rangle^2}{(1-hD)^2}\right]$$

$\langle \sigma^\mu \rangle^+$ and $\langle \sigma^\mu \rangle^-$ respectively denote the positive and negative parts of the stress tensor (in terms of principal values), $\langle x \rangle$ stands for the positive part of the scalar x, $\langle x \rangle$=max(x,0) (for metals $h\approx0.2$ [6,7]. In Fig.2, different values of *h* are plotted in a Haigh diagram [8] ($\sigma_{alt}=f(\sigma_m)$ with $\sigma_{alt}=0.5\,\sigma_{max}\,(1-R_\sigma)$) for a steel at 10^7 cycles. If *h*=1, there is no mean stress effect, and if *h*=0.2, there is a difference between positive and negative mean stress. Comparing the Goodman line (with the coordinates (0, σ_f), (σ_u, 0)) [9] valid for unixial loadings (smooth specimen) we can see a gap between lines with *h* and the Goodman line for high stress ratios. We can experimentally observe that in function of the stress ratio level, the stress confinement in the loading (or plastified) zone and the multiaxiality of local stresses, the parameter *h* could not be sufficient to correctly represent real behavior of materials. So, it could be interesting, to modify the model in order to be more realistic in some cases. Next paragraph deals with this modification.

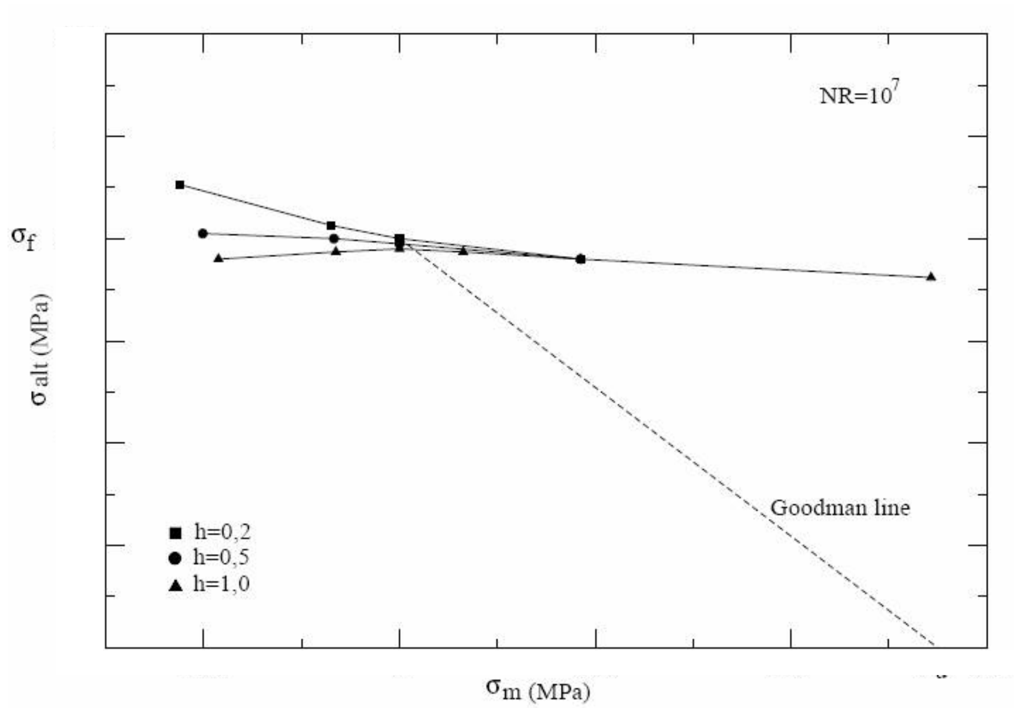

Figure 2 - Influence of the micro-defects closure parameter *h* on a Haigh diagram.

Consideration of the first stress invariant

In geomaterials, it is usual to introduce the Drucker-Prager yield criterion [10] (first stress invariant in the yield function) to simulate cohesive properties of materials. Assuming here that the imbedded inclusion has a confined behavior, the yield function at microscale is so modified by adding the first stress invariant:

$$f^{\mu} = (\tilde{\boldsymbol{\sigma}}^{\mu} - \boldsymbol{X}^{\mu})_{eq} + kTr\tilde{\boldsymbol{\sigma}}^{\mu} - \sigma_f^{\infty}$$

with *Tr(.)* the first stress invariant of the stress tensor at micro scale, and *k* a Drucker-Prager parameter to identify.

Influence of the parameter k

By time integration of the constitutive equations for a proportional loading in tension-compression, we obtain the expression of the number of cycles to rupture $N_R=N_R(D=D_c)$ when the damage D reaches the critical damage D_c. The denominator of this expression can be written as $\Delta\sigma-2\sigma_f^{\infty}+2k\sigma_m$. The asymptotic fatigue limit corresponding to an infinite number of cycles to rupture is obtained for $\Delta\sigma-2\sigma_f^{\infty}+2k\sigma_m=0$ and corresponds to an asymptote of the Wöhler curve. By keeping in mind the definition of a Haigh diagram ($\sigma_{alt}=f(\sigma_m)$) we obtain the expression of the asymptotic fatigue limit: $\sigma_{alt}=\sigma_f^{\infty}-k\sigma_m$. In a Haigh diagram, *k* is the slope of the asymptotic limit which is the border between the no crack initiation and the crack initiation domain.

EQUIVALENCE BETWEEN DRUCKER-PRAGER CRITERION & SINES CRITERION

The Sines fatigue criterion is based on phenomenological considerations and uses the octahedral shear A_{II} and the mean value of the hydrostatic stress. The Sines criterion predicts the appearance of a crack initiation when the equation below is satisfied :

$$A_{II} - \sigma_f(1 - 3b\,\bar{\sigma}_H) = 0$$

where :

$$A_{II} = \frac{1}{2}\sqrt{\frac{3}{2}(\sigma_{ij\ max}^{D} - \sigma_{ij\ min}^{D})(\sigma_{ij\ max}^{D} - \sigma_{ij\ min}^{D})} = \frac{\Delta\sigma_{\Sigma}}{2}$$

with S_{ij}^{D} the components of the stress tensor deviator.

The modified yield function previously introduced gives at yield limit for the maximum and minimum loadings (under proportionnal loadings) :

$$\begin{cases} -(\sigma_{min})_{eq} - \sigma_f + k\sigma_{min} = 0 & (1) \\ (\sigma_{max})_{eq} - \sigma_f + k\sigma_{max} = 0 & (2) \end{cases}$$

From these equations we rapidly obtain :

$$\frac{\Delta\sigma_{eq}}{2} - \sigma_f + k\bar{\sigma} = 0 \qquad ((1)+(2))/2$$

This equation can easily be compared to the Sines criterion one :

$$\frac{\Delta\sigma_{eq}}{2} - \sigma_f + 3b\sigma_f\bar{\sigma_H} = 0$$

We have $\bar{\sigma} = 3\bar{\sigma}_H$ so we found $k = b\sigma_f$

Thus, the modification of the yield function at microscale leads to obtain the Sines initiation criterion.

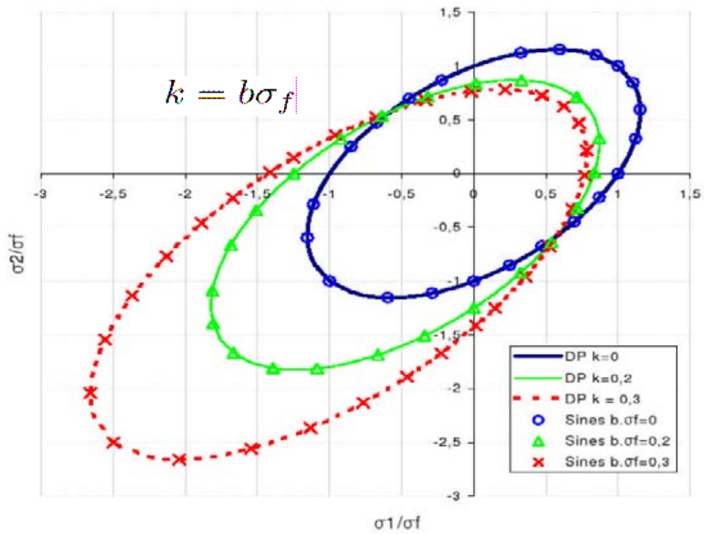

Figure 3 - Ellipsis obtained for Sines criterion and for the 2 scale model modified yield function, for biaxial loadings.

FATIGUE POST-PROCESSOR

The corresponding modifications of the yield function have been implemented in the fatigue post-processor DAMAGE [11].
This post-processor [12] is a Fortran program which explicitly solves the two scale damage model constitutive equations with an Euler backward scheme. A graphical interface makes the software use quite easy. For a given material parameters file and for a given loading sequence, the program calculates the time to crack initiation, i.e. the time to reach the critical damage D_c. The inputs are either 1D or come from any 3D Finite Element computations.

Material parameters identification

For the material parameters identification, the following procedure may be used (applicable for smooth specimen to fit the Goodman expression).

1. The mesoscale parameters (E, ν, α, σ_y, C_y) are identified at each temperature on the monotonic tensile curve.
2. Parameters *h* and D_c, the default constant values for metals, are taken equal to *h*=0.2 and D_c=0.3.
3. The identification of the asymptotic fatigue limit and of the *k* parameter is related to and depends on the stress ratio of the experimental Wöhler curve used.
   - If the stress ratio $R_\sigma=\sigma_{min}/\sigma_{max}=-1$: the asymptotic fatigue limit is the experimental “asymptote” at very high number of cycles. Then, *k* is defined as the Goodman expression: $k= \sigma_f^\infty / \sigma_u$
   - If the stress ratio $R_\sigma=\sigma_{min}/\sigma_{max}\neq-1$: the asymptote $\sigma_{max}= \sigma_{max}^\infty$ gives the material parameter fatigue limit σ_f^∞ as:
     $\sigma_f^\infty=1/2\ \sigma_{max}^\infty\ [(1- R_\sigma)+k(1+ R_\sigma)]$
     By using the Goodman expression for *k*, the previous equation leads to:
     $\sigma_f^\infty = (0,5\ \sigma_{max}^\infty\ (1- R_\sigma)\ \sigma_u) / (\sigma_u - (1+ R_\sigma)\ 0,5\ \sigma_{max}^\infty\ )$
     Then, *k* can be determined.

4. The damage parameters S and s are pre-identified from a nonlinear fitting (automatic in DAMAGE software): the experimental Wöhler curve and the approximate closed-form solution for the number of cycles to rupture is used to automatically obtain a set of parameters S,s.
5. The parameter S is finally adjusted by comparison with the reference curve but, this time, by using DAMAGE to compute the Wöhler curve instead of the approximate formula.

Mean stress effect with the fatigue post-processor DAMAGE

A Haigh diagram is computed using the post-processor DAMAGE for smooth specimen (Fig.3) (note that for notched specimen, due to confined stresses and strains and higher multiaxiality rate, *k* must be adapted and chosen lower than the one of smooth specimen). Cases *k*=0 and *k*≠0 are compared. Asymptotic fatigue limit (bold lines) and lines at NR=10^6 cycles (thin lines) are plotted.
We can observe that:
- for *k*=0, the asymptotic limit in the Haigh diagram is a horizontal line. For a high mean stress loading, corresponding to a point below this asymptotic limit, the post processor will always find an infinite lifetime.
- For *k*≠0, the asymptotic limit in the Haigh diagram is a line with a negative slope. This property allows for the points corresponding to high mean stresses to have a finite lifetime.

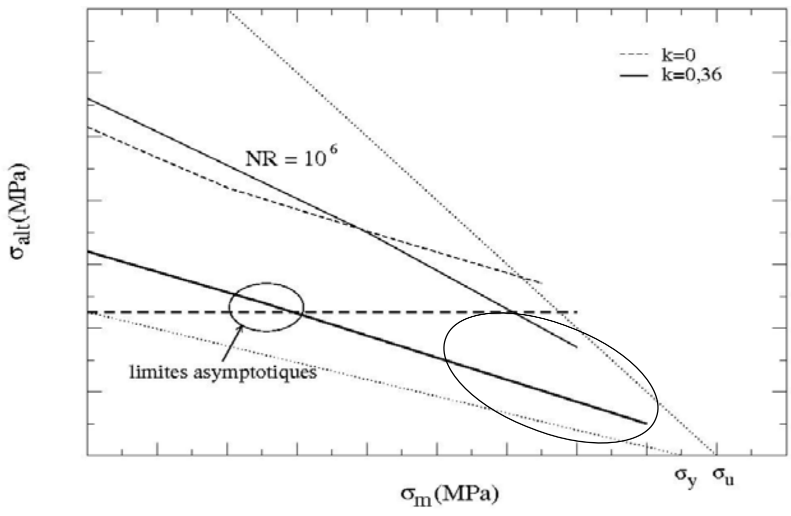

Figure 4 - Influence of *k* in a Haigh diagram

BIAXIAL FATIGUE TESTS

Experimental set up

Biaxial tests on thinned Maltese cross shaped specimen have been conducted in order to test the two scale model prediction performances. Two materials have been tested : 304L (for nuclear applications) and a titanium alloy (for space applications). We will only discuss here the tests on the titanium alloy.

A thinned Maltese cross specimen has been designed (Fig.1) thanks to Finite Element Cast3m code and DAMAGE post-processor. This specimen is made to initiate HCF crack on its central area under various (proportional, non proportional) biaxial loadings. The tests are carried out on a multiaxial testing machine (ASTREE) at LMT Cachan. This testing machine has six servohydraulic actuators. The four horizontal actuators used herein have a 100 kN load capacity and a 250 mm stroke range.

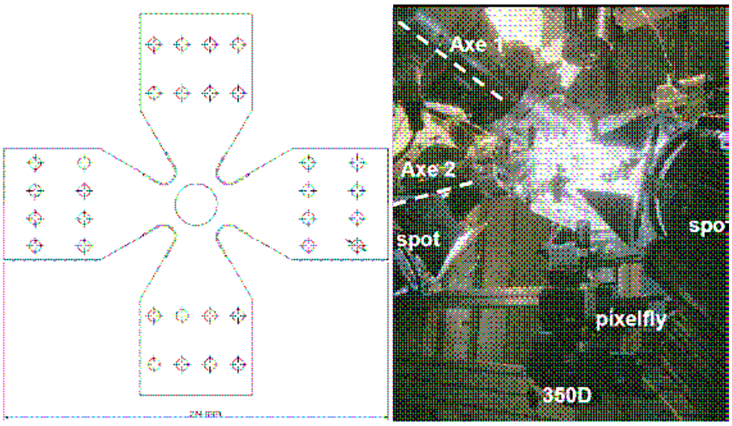

Figure 5. a) Geometry of the specimen. b) General view of a biaxial test.

Biaxial tests are load controlled and are performed at room temperature.

The test frequency is 20Hz. Even if the permanent strains in central area are very small, digital image correlation was used to have the strains fields in the entire specimen and not only on a small zone corresponding to a gauge.

Strain fields are monitored using different cameras. Eight biaxial tests were performed, different kinds of experimental set up were tested.

Eight tests are presented here (represented by a letter going form A to H). Tests could be divided into three kinds of various loadings :

Tests A and B correspond to equibiaxial tests (in phase) at two different load ratios. (R_F=0.1 for test A, R_F=0.5 for test B).

Test C, D, E, F and G are non equibiaxial tests (in phase) with different load ratios on each direction.

test	C, D	E	F	G
R_F direction 1	0.5	0.6	0.4	0.1
R_F direction 2	0.4	0.5	0.3	1.0 (F=cste)

Table 1 – Summary of tests

Test H corresponds to 90° out of phase non equibiaxial test with different load ratio on each direction.

As previously described, the maximum strain amplitude loading is defined to compare biaxial and uniaxial tests. Figure 6 represents fatigue results of biaxial tests and uniaxial tests performed at a load ratio of R_F=0.1 (test C presented a machining irregularity in the central area which led to a shorter life than expected).

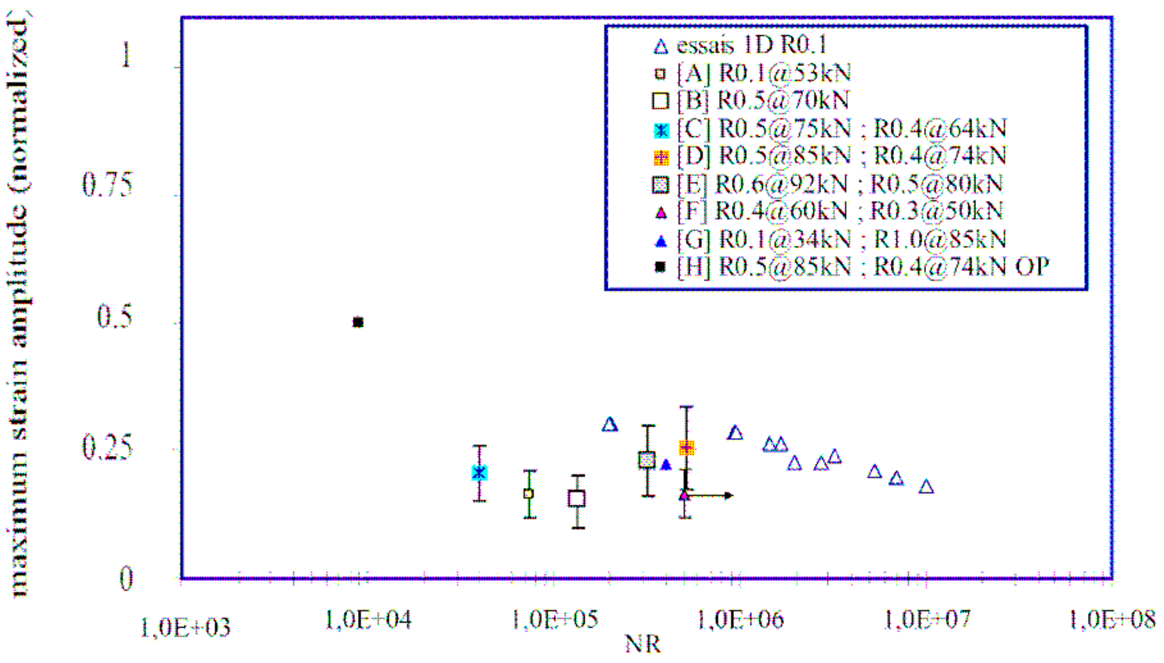

Figure 6. : Summary of biaxial fatigue results for titanium alloy (A, B, C and F tests specimen geometries are thicker).

As all these tests were performed at different load ratios, with different applied forces, it is quite difficult to compare them. Anyway, with the proposed representation of results, one can observe that :

- D versus H tests : out of phase effect under the same loading. For out of phase loading, the maximum strain amplitude is about twice as the in-phase corresponding loading. The number of cycles for the out of phase loading is 40 times less than the in-phase loading.

One has to remember that each biaxial test (i.e. each loading configuration) was performed just once. So, the remarks presented here are made without taking eventual scatter of results into account. Anyway, as it can be seen in uniaxial fatigue results, scatter in this material is rather small.

The comparison between the predicted and the experimental number of cycles is shown here-after :

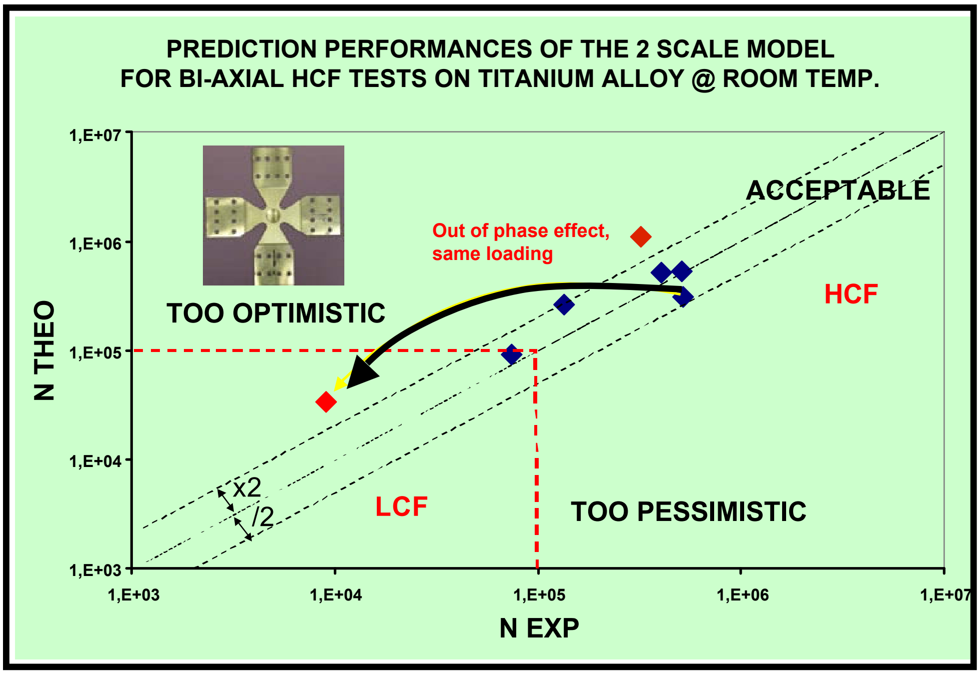

Figure 7. : Comparison between predicted and experimental values for titanium alloy.

CONCLUSION

The introduction of a Drucker-Prager term in the yield function of the LMT two scale damage model makes the model and the corresponding post-processor DAMAGE able to better take into account the mean stress effects in a wide stress ratios range. The identification of the Drucker-Prager parameter is simple, and is function of the stress ratio of the experimental Wöhler curve. This parameter k has a physical meaning : it is the slope of the asymptotic limit in a Haigh diagram. Thanks to this parameter, it is now possible to more precisely represent the behavior of real specimen under HCF loadings.
Moreover, it has been established that this modification of the yield function at microscale leads to obtain a crack initiation criterion equivalent to the Sines one.
The two scale damage model was used herein to calculate the NR of biaxial fatigue tests performed at LMT-Cachan on a titanium alloy. A factor 2 can globally be observed between experimental and numerical number of cycles to rupture.

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