# MATHEMATICAL MODEL FOR ESTIMATION OF PARAMETERS OF DISTURBANCES THAT STABILIZE UNSTABLE PROCESSES

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There exists a lot of physical systems that are unstable with respect to development of inner unsteady processes. A great number of publications have arisen recently that are devoted to ways for stabilizing such systems including methods using external harmonic perturbation of certain frequency (for example, [1,4]).

The present work addresses to development of the method for estimation of some properties of such stabilizing disturbances on the basis of previously developed method for description of discontinuous functions.

Works [2–3] concern the method of description of jump processes via functions having stepwise discontinuities. This approach permits to depict this functions' behavior in transition zones and the length of transition zone. This method is based on utilization of additional parameter $\eta$ having a form of symmetric unite function with shifted coordinate origin:

$$\eta = U_{-}(x-\varphi_2) = \begin{cases} 0, \text{ for\_} x < \varphi_2 \\ 1, \text{ for\_} x \geq \varphi_2 \end{cases}$$

and, also, on the using of noncommuting operators, the action of first one corresponds to a shift along considered function, while the action of second one corresponds to multiplication by the introduced additional variable.

Let the physical phenomenon exists that is described by a curve located in the plane with coordinates $(\alpha, m)$. At the piece located between two points $\alpha_1$ and $\alpha_2$ the function have a rapid (jump) transformation (Fig. 1). $\alpha_1$ is located before transition point, while $\alpha_2$ coincides with the transition point. Let us describe

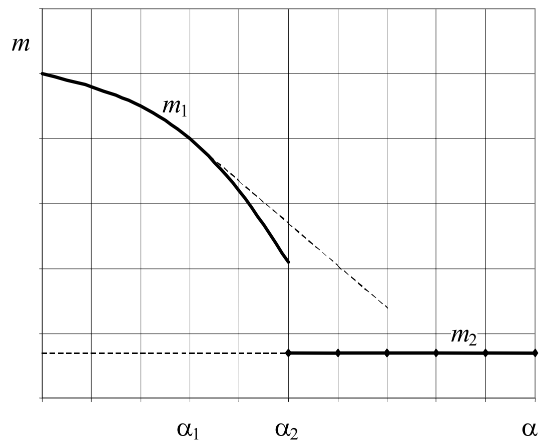


Fig. 1

the behavior of function in some direction in vicinity of transition point. The mean value of square of function $m$ on this piece is determined by expression:

$$\langle m^2 \rangle = \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} m^2 d\alpha .$$

This expression may be recast in another form: $\int_{\varphi_1}^{\varphi_2} M^2 d\varphi = 1$. Herein: $M^2 = \frac{m^2}{\langle m^2 \rangle}$, $\varphi = \frac{\alpha}{\Delta\alpha}$, $\Delta\alpha = \alpha_2 - \alpha_1$.

As the mathematical model of function in zone of jump transition we shall use a step discontinuity function. Let us note a part of function before break point as $m_1(\alpha)$ and a part of function past break point as $m_2(\alpha)$. Let us extrapolate both parts on all region of by linear functions, so that the branch $m_1(\alpha)$ past point $\alpha_1$ (in transition zone and past it) was a line tangent to the branch $m_1$ at point $\alpha_1$. Similarly, the branch $m_2(\alpha)$ before point $\alpha_2$ (in transition zone and before it) should be a line tangent to branch $m_2$ at $\alpha_2$. The break function may be described using additional parameter: a function $\eta$ that equals unit everywhere before breakpoint and equals zero everywhere past the breakpoint. Additional variable may have an appearance:

$$\eta = \frac{m(\alpha) - m_2(\alpha)}{m_1(\alpha) - m_2(\alpha)} \equiv \frac{M - M_2}{M_1 - M_2} =$$

$$= \frac{\frac{\partial m(\alpha)}{\partial\alpha} - \frac{\partial m_2(\alpha)}{\partial\alpha}}{\frac{\partial m_1(\alpha)}{\partial\alpha} - \frac{\partial m_2(\alpha)}{\partial\alpha}} \equiv \frac{\frac{\partial M}{\partial\varphi} - \frac{\partial M_2}{\partial\varphi}}{\frac{\partial M_1}{\partial\varphi} - \frac{\partial M_2}{\partial\varphi}},$$

where the function $m(\alpha)$ equals $m_1(\alpha)$ for $\alpha < \alpha_2$ and $m_2(\alpha)$ for $\alpha \geq \alpha_2$. The derivative of $\eta$ over $\varphi$ is equal to

$$\frac{\partial\eta}{\partial\varphi} = -\frac{\partial\eta}{\partial(-(\varphi - \varphi_2))} =$$

$$= -\delta(-(\varphi - \varphi_2)) = -\delta(\varphi - \varphi_2),$$

where $\delta(\varphi)$ – Dirac delta function.

Using the above expressions one may determine the mean value of the square of the function $m$ in the space with additional variable:

$$\langle m^2 \rangle = m_1 m_2 \varphi_2 - m_1^2 \varphi_1 + m_2 \frac{\partial m_1}{\partial\varphi} \varphi_2 =$$

$$= \frac{\alpha_2}{\Delta\alpha} m_1 m_2 - \frac{\alpha_1}{\Delta\alpha} m_1^2 + \alpha_2 m_2 \frac{\partial m_1}{\partial\alpha} .$$

Here, for brevity, the notations are omitted that show the magnitude of function $m_2$ is selected at point $\alpha = \alpha_2$, and a magnitude of $m_1$ at the point $\alpha = \alpha_1$.

The above expression demonstrate that the mean value of function at point $\alpha_1$ is chosen as the value on branch $m_1$ while at point $\alpha_2$ we have a «mixed» state that is some averaged between the value on the branch $m_2$ and extrapolated magnitude of function on the branch $m_1$. This asymmetry determines the choice in notation of first and second branches: $m_1(\alpha)$ should describe the state of system that is stable with respect to inner processes.

The transition of unstable physical process from one mode to another is most probable in that case, if the probabilities of system occurring in two states $m_1(\alpha)$ and $m_2(\alpha)$ are equal, that is two branches of function $m_1(\alpha)$ and $m_2(\alpha)$ satisfy the dispersion relation in transition zone.

Let us consider the commutational relation of two operators in order to find the dispersion relation for break region: $\hat{k} = i\frac{\partial}{\partial\varphi}$ и $\hat{\eta} = \eta$:

$$\left[\hat{k}; \hat{\eta}\right] M = -iM\delta(\varphi - \varphi_2).$$

Using expressions:

$$\Delta_M^k = \left[ \left( \hat{k}^2 M, M \right) - \left( \hat{k} M, M \right)^2 \right]^{1/2},$$

$$\Delta_M^\eta = \left[ \left( \hat{\eta}^2 M, M \right) - \left( \hat{\eta} M, M \right)^2 \right]^{1/2},$$

$$\Delta_M^k \Delta_M^\eta \sim \frac{1}{2} \left| \left( \left[ \hat{k}; \hat{\eta} \right] M, M \right) \right|,$$

where the expression in circle brackets are the scalar products in Hilbert space, we obtain the following expressions:

$$\Delta_M^k = \left( \frac{1}{2} M_2 \left( \frac{\partial M_1}{\partial \varphi} - \frac{\partial M_2}{\partial \varphi} \right) + \frac{1}{4} M_2^2 \left( M_1 - M_2 + \frac{\partial M_1}{\partial \varphi} \right)^2 \right)^{\frac{1}{2}},$$

$$\Delta_M^\eta = i \left( M_1^2 \varphi_1 + \left( \frac{1}{2} M_2^2 \varphi_2 - M_1^2 \varphi_1 \right)^2 \right)^{1/2}.$$

In order the density of probability of belonging to one of branches were equal in transition zone, it is necessary that the dispersion of order parameter module was approximately equal to 0.5, that is $\Delta_M^\eta \sim i/2$. By imposing this condition one may obtain the functional dependence of $\varphi_2$ on $M_1$ and $M_2$, or $\Delta\alpha$ on $m_1$, $m_2$ and $\alpha_2$:

$$\pm \left( \left( \frac{\alpha_2}{\Delta\alpha} \right)^2 \frac{m_2^4}{4} - \frac{\alpha_2}{\Delta\alpha} \left( 1 - \frac{\alpha_2}{\Delta\alpha} \right) m_1^2 m_2 \times \left( m_1 + \frac{\partial m_1}{\partial \alpha} \Delta\alpha - m_2 \right) \right)^{\frac{1}{2}} \Bigg/ \left( \frac{\alpha_2}{\Delta\alpha} \left( m_1 m_2 - m_1^2 + m_2 \frac{\partial m_1}{\partial \alpha} \Delta\alpha \right) + m_1^2 \right) \sim \frac{1}{2}. \quad (1)$$

This dependence may be used for estimation of transition zone duration, that is for estimation of magnitude of $\Delta\alpha$ that separates the physical system, situated at point $\alpha_1$ from point of transition $\alpha_2$, if above the current parameters of function $m_1$ and its derivative we know the supposed properties of transition point, i.e. the value of $m_2$.

As this take place, the dispersion relation in dimensional variables takes a form:

$$(\Delta\alpha)^2 \left( \frac{\partial m_1}{\partial \alpha} \right)^2 + 2\Delta\alpha \frac{\partial m_1}{\partial \alpha} \left( m_1 - m_2 + \left( \frac{m_1^2}{m_2} \frac{1}{\partial m_1 / \partial \alpha} + \alpha_2 \right) \left( \frac{\partial m_1}{\partial \alpha} - \frac{\partial m_2}{\partial \alpha} \right) \right) + 2\alpha_2 m_1 \left( 1 - \frac{m_1}{m_2} \right) \left( \frac{\partial m_1}{\partial \alpha} - \frac{\partial m_2}{\partial \alpha} \right) + m_1 \left( m_1 - 2m_2 \right) = 0, \quad (2)$$

Here the substitution was performed: $\alpha_1 = \alpha_2 - \Delta\alpha$.

Equations (1)-(2) are written for transit zone. For the sake of brevity, the notations are omitted that indicate the value of function $m_2$ to be taken at point $\alpha = \alpha_2$ while the value of function $m_1$ at point $\alpha = \alpha_1$.

The considered system is supposed to have regions stable to inner unsteady processes and regions having no such stability. The properties of stabilizing disturbances should be determined by parameters of regions, which are unstable with respect to inner processes. So the development of the methods for determination of these parameters is the main purpose of present work.

Let us consider the process of high velocity jet impingement to the disc (Fig. 2). In accordance with [4] in results of such impingement the temporally growing modes of disturbances appear which cause the instability of disc. These instabilities may be compensated by hollows situated on the disc surface from the impinging jet side. The specific dimension

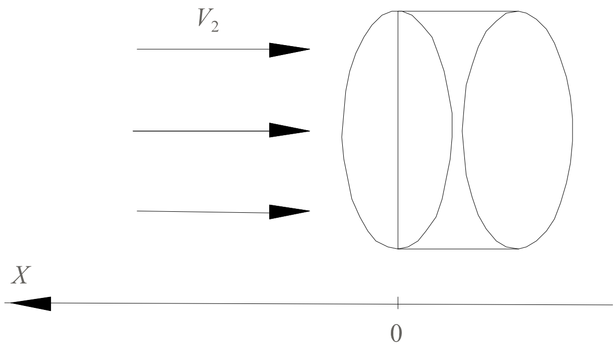


Fig. 2

of these hollows corresponds to the wavelength of most rapidly increasing disturbances.

Herein, we attempt theoretically (using the method of discontinuous functions) to determine the specific dimension of disperse structure on the surface of disc that suppers the development of disturbances causing instability of process.

Let $V(x)$ is the velocity of approaching flow at point *x*. *X* axis is parallel to gas flow.

The mean module of velocity gradient between $x_1$ and $x_2$ may be expressed as:

$$\left\langle \frac{\partial V}{\partial x} \right\rangle = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \frac{\partial V}{\partial x} dx .$$

After transition to variables used in discontinuous functions method, this expression may be recast as:

$$\left\langle m^2 \right\rangle = \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} m^2 d\alpha ,$$

where $\alpha = x$, $m = \sqrt{\frac{\partial V}{\partial x}}$.

According [2–3], the values of functions and their derivatives of “1” and “2” in expressions (1)–(2) cannot be chosen arbitrary because expression (1) is not invariant to the choice of frame origin. It occurs due to the circumstance that the physical phenomena specified by jump transitions are always determined by certain own scales of variable parameter α, for example, by specific values of Re number at the transfer from a laminar to the turbulent mode. As this take place, the zero on the abscissa axes should be chosen so that ensure the conservation of process direction at the change of the sign of parameter α relatively parameter η.

From these considerations, in the present problem the index “1” should be attributed to the values of function and its derivative near the disc surface while the index “2” corresponds the distant values.

Under this choice of parameters: $\alpha_1 = 0$, $\Delta\alpha = \alpha_2$, $m_1 = \sqrt{\frac{\partial V}{\partial x}}\Big|_{x=0} = \text{const}$, $\frac{\partial m_1}{\partial \alpha} = 0$.

By substituting these expressions to (1) we get: $m_1 m_2 \sim \pm m_2^2$. This engenders one trivial solution: $m_2 \sim 0 \ \Rightarrow V_2 = const$ (condition of unstablations absence) and two nontrivial solutions: $m_1 \sim \pm m_2$.

Because $m_1 \geq 0$ and $m_2 \geq 0$ exist only one nontrivial solutions: $m_1 \sim m_2$.

Let us consider latter solution. From (2) we deduce that if $m_1 \sim m_2$, then

$$\frac{\partial m_2}{\partial \alpha} = -\frac{1}{2} \frac{m_1}{\alpha_2} .$$

It should be recalled that the magnitude of derivative at point $\alpha_2$ was determined here. It is impossible to determine precisely the functional form of $m_2(\alpha)$ from this expression. However, due to linear extrapolation of function $m_2(\alpha)$ in transition zone (that well operates in simplest cases of consideration of small ranges of α), with the account of condition $m_2 \sim m_1$, for $\alpha \to \alpha_2$, the following dependence may be obtained:

$$m_2 = \frac{m_1}{2\alpha_2}(3\alpha_2 - \alpha).$$

This expression may be rewritten as:

$$\frac{\partial V_2}{\partial x} = \frac{m_1^2}{4x_2^2}\left(x - 3x_2\right)^2,$$

where $m_1 = const$.

Because the pressure in the boundary layer is constant, the velocity gradient may be considered constant over all this region. Thus $m_1^2 = \frac{\partial V_1}{\partial x} \approx \frac{V_0}{x_2}$.

By resolving this equation with the account of boundary condition $V_2\big|_{x\to x_2} \approx V_0$, we obtain:

$$V_2 = \frac{5}{3}V_0 + \frac{1}{12}V_0\left(\frac{x}{x_2} - 3\right)^3. \qquad (3)$$

Coordinate $x_2$ determines the thickness of the boundary layer at which edge the velocity gradient changes. It should be calculated from concrete conditions of problem.

It can be seen from this expression that $V_2 = 0$ if $x_0 \approx 1/3\, x_2$. It is the difference between this coordinate and coordinate $x_1 = 0$ where velocity $V_1 = 0$: $\Delta = x_0 - x_1 \approx 1/3\, x_2$ that gives the specific scale of discrete structure on the disc surface which should stabilize the arising disturbances.

In first approximation the boundary layer thickness for laminar mode may be taken as equal to square root from inverse Reynolds number, that is $x_2 \sim R/\sqrt{\mathrm{Re}}$, where R is the disc radius, so $\Delta \approx \frac{1}{3}\frac{R}{\sqrt{\mathrm{Re}}}$ (Fig. 3).

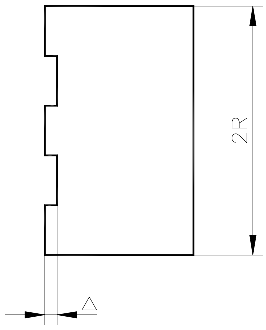


Fig. 3

At the posing of discrete structure with the calculated reference scale on the disc surface, the transversal dimension may change with time, because it can not be considered as determined in frames of considered problem statement. In cumulative effect approach [4] it will depend on the elastic properties of the material above longitudinal reference scale.

## References


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