MULTI-LEVEL FINITE ELEMENT / EXTENDED FINITE ELEMENT APPROACHES FOR CRACK PROPAGATION IN INDUSTRIAL FATIGUE ANALYSIS PROCEDURES

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The eXtended Finite Element Method (XFEM) [1] is a numerical method which handles geometries containing discontinuities without the need of building a conforming mesh. In addition, these discontinuities are introduced using the Level Set (LS) Method [2]. These two techniques used together are particularly suited for application to fracture mechanics problems involving the introduction of 3D cracks in complex geometries [3].

However, the XFE and LS methods have not been implemented yet in general-purpose commercial Finite Elements (FE) software due to several theoretical and implementation related problems. In the present work, two substructuring methods are investigated in order to allow for the use of the XFEM within commercial FE codes without the need for modifying their kernel.

In this approach, the global FE problem is decomposed into two subdomains, the “safe domain” and the “cracked domain”, based on the value of the Level Sets representing the crack. The host FE-software treats the “safe domain” while an independent XFE-code treats the “cracked domain”.

The first substructuring method consists in calculating the Schur complement matrix of a cracked super-element with the XFE-code. The second technique introduces the Finite Element Tearing and Interconnecting Method (FETI) [4] which ensures the compatibility of the displacements at the interface between the cracked and safe subdomains. The stiffness matrices and nodal forces are provided by the XFE and FE codes for the cracked and safe subdomains, respectively. The solutions obtained with these two techniques are rigorously equivalent to those computed with the stand-alone XFE-code. First, the computational efficiency of the two approaches is demonstrated. Second, a validation is proposed towards comparison with reference values of the stress intensity factors in simple 3D cracked geometries.

Finally, this contribution presents two applications of the FE-XFE-FETI method:

- A 3D crack analysis inside a hydraulic cylinder submitted to an internal pressure.
- A crack analysis on the outer skin of a structural part modeled with shells. A local plane stress formulation is defined on the

cracked subdomain under the assumption that out-of-plane loading can be neglected.

Theoretical background

Crack representation by the level set method

The Level Set method was introduced by Sethian [2] to model moving interfaces. A surface is represented by the zero level set of a function. Two "level sets" are needed for modelling a crack, the normal $\psi_n(x)$ and the tangent $\psi_t(x)$ "level sets" (see Fig. 1).

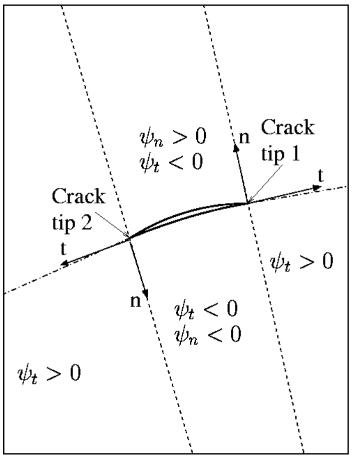

Fig. 1

With this approach, discontinuities and/or singularities can be added without modifying the meshes or even the geometry.

The eXtended Finite Element Method

As the eXtended Finite Element Method is based on the Partition of Unity Method [5], the a priori knowledge of the solution in a region of the modelled sample will lead to the definition of enrichment functions. Two types of enrichment functions are considered for fracture mechanics problems:

- A Heaviside (or step) function, in order to model the discontinuity of the displacement field:

$$H(x) = \begin{cases} -1 & if \quad \psi_n < 0 \\ +1 & if \quad \psi_n > 0 \end{cases}.$$

- The singular term of the linear elastic fracture mechanics (LEFM) [6] displacement field expansion, in order to improve the accuracy of the solution at the crack tip:

$$\{F^l\} \equiv \{ \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right),$$
$$\sqrt{r}\sin\left(\frac{\theta}{2}\right)\cos(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \},$$

where (r, θ) is the local polar coordinates system for a reference frame attached to the crack tip.

Finally, the displacement field becomes in the extended finite element formulation:

$$u(x) = \sum_{i \in I} u_i \Phi_i(x) + \sum_{j \in J} b_j H(x) \Phi_j(x)$$
$$+ \sum_{k \in K} \Phi_k(x) \sum_{l=1}^{4} c_k^l F^l(x)$$

where I is the set of all nodes of the domain, J is the set of nodes whose support is cut by a crack (circle nodes on Figure (2)), K is the set of nodes containing the first and the second crack tip (square nodes on Figure (2)), u_i are the usual degrees of freedom (i.e. displacement) for node i, Φ_i is the shape function associated to node i, b_j is the jump in the displacement field across the crack at node j (if the crack is aligned with the mesh, b_j represents the opening of the crack) and c^l_k is the additional degrees of freedom associated with the crack tip enrichment functions.

Only a set of nodes can be enriched. If every node of the mesh is enriched, linear de-

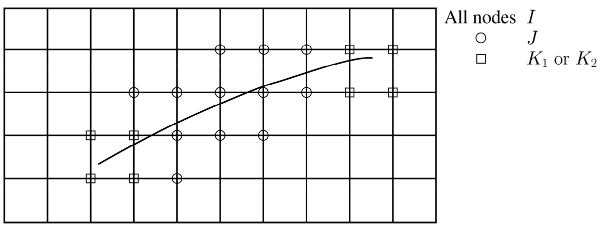

Fig. 2

pendency between equations occurs and the global stiffness matrix becomes singular.

Substructuring FE-XFE approaches

Decomposition of the domain

As a prerequisite to the use of the substructuring methods, we need to proceed to the decomposition of the domain. A crack is introduced in a finite element mesh by means of “level sets”. The mesh is decomposed into two subdomains, a safe subdomain handled by the FE-software and a cracked subdomain handled by the XFE-code (see Fig. 3).

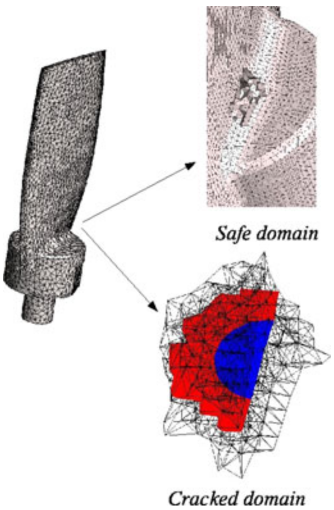

Fig. 3

Super-Element

The coupling of a FE-software and an XFE-code by using a super-element was first proposed by Bordas in [7]. The method consists in decomposing the problem into a safe domain and one or several cracked domains and in treating either the safe or the cracked domain as a super-element. In this case, the XFE-code is used to build the Schur complement of the cracked domain. We assumed that no forces are applied neither on the boundary nor in the interior of the cracked domain. This assumption is required by the super-element format used (the PERMAS format available in SAMCEF™ was chosen for the first simple applications). This leads to the following expression of the Schur complement:

$$K_{Schur} = K_{bb} - K_{ib}^{T} K_{ii} K_{ib}$$

where K_{ii} is the stiffness linking the dof’s interior of the cracked domain, K_{bb} is the stiffness matrix linking the boundary dof’s and K_{ib}.

The Schur complement of a substructure is generally dense. First, it is of the uttermost importance to reduce the memory needed for- and the computational cost associated with the computation of the Schur complement. Second, it is recommended to limit the size of the Schur complement as it has a dramatic effect on the sparsity of the global system.

To minimize these effects, the domain decomposition is defined for the computation of the Schur complement such that the first layer of non-enriched nodes composes the boundary of the cracked domain.

In order to compute stress intensity factors at the crack tip, a new domain is defined such that it contains the contour of the equivalent domain integral [8]. Then a XFE problem is solved on this domain with the result computed by the FE-software as Dirichlet boundary conditions. Finally, the stress intensity factors are computed using the equivalent domain integral.

Coupled FE/XFE based on the FETI method

The Finite Element Tearing and Interconnecting method (FETI) is a substructuring method. FETI has been developed to handle and solve large-scale problems. FETI allows for both parallel and sequential computation schemes.

The whole domain is divided into several substructures; the problem to be solved is reduced to an interface problem.

The stiffness matrices and the nodal forces are extracted from the subdomains. The global system is solved by means of the FETI method. For details see Farhat and al. [4].

Numerical validation

Description of the sample

A 3D bar with an inclined crack is studied for various misorientation angles. The stress intensity factors are computed along the crack front and compared with those obtained with a Meshless-code (note that there are no analytical solutions available for this crack configuration) described in [9].

The sample is shown in Fig. 4. The upper face is submitted to a tensile stress and the bottom face is fixed in the *Z*-direction. To avoid singularity of the global matrix, one edge parallel with the *X*-axis and one edge parallel with the *Y*-axis are fixed in the *Y*-direction and *Z*-direction, respectively.

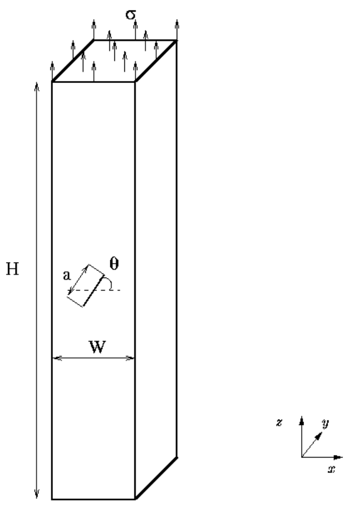

Fig. 4

- $a/W = 0.2$,
- $H/W > 2$,
- $\sigma = 1.0$ Pa,
- θ varies from 0° to 75°.

Numerical result

We treat the FETI and the super-element coupling methods together, using the same mesh. The computed solutions are rigorously identical and obviously identical to the reference XFE solution. Indeed, the same global matrix is solved and only the solver accuracy can influence the quality of the solution.

The arithmetic mean values of the stress intensity factors along the crack fronts are compared with those coming from the Meshless-code. The values of the stress intensity factors near the external boundaries of the sample are not taken into account because the domain of the integral is not large enough.

Fig. 5 shows that the mean values of K_{II} are overestimated for low value of θ. This is due to spurious stress concentration induced by the XFEM, (this is not related to the substructuring method). These spurious stress concentrations generate local erroneous stress intensity factors.

If these erroneous values of K_{II} are eliminated (by hand), the agreement with the reference values is good (see Fig. 6). The maximum relative difference between the values obtained with the substructuring method and the reference values is less than 5%.

As the number of elements involving a spurious stress is relatively low, refining the mesh in the vicinity of the crack tip allows for an accurate computation of the global stress intensity factors. Indeed the volume of the

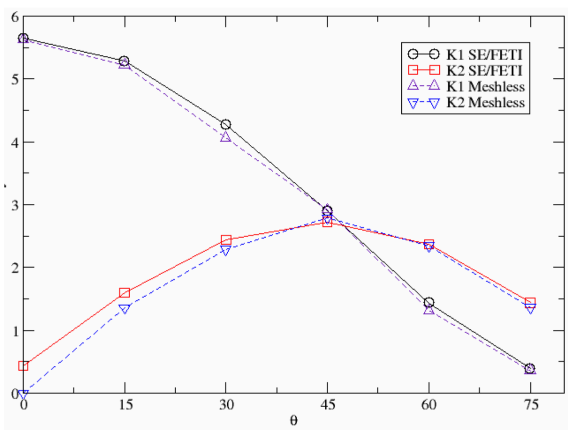

Fig. 5

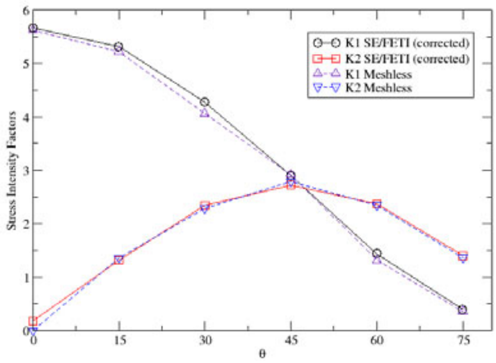

Fig. 6

concerned elements becomes small compared to the domain of the equivalent domain integral. However, this solution did not fix the problem of spurious stress and, locally, the stress intensity factor can still be erroneous.

Compared to the FETI method for a problem that involves about 100000 dof's, the computational time for the super-element method is three to four times bigger and the maximum allocatable memory is used. The use of the super-element method is more judicious for small crack problems.

Applications

Hydraulic cylinder under internal pressure

We apply the method to the computation of the stress intensity factors along the front of a crack located inside a hydraulic cylinder under internal pressure. This study was realized in the framework of the European Commission Integrated Project PROHIPP (New design and manufacturing processes for high pressure fluid power products). The hoop stress near the oil port is detailed in Fig. 7.

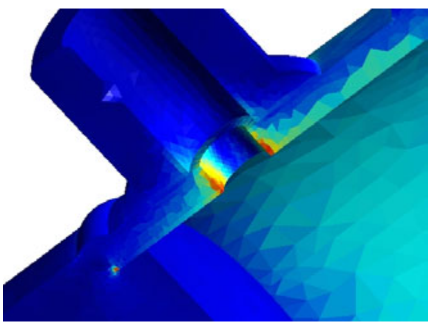

Fig. 7

Indeed, the maximum stress is located in the plane of symmetry where a crack has been experimentally observed.

As a first approximation, we focus on the part of the cylinder around the oil port where we compute the stress intensity factors. This zoom is required since the mesh of the whole cylinder is far too coarse for a small crack analysis. Fig. 8 shows the model used for the computation.

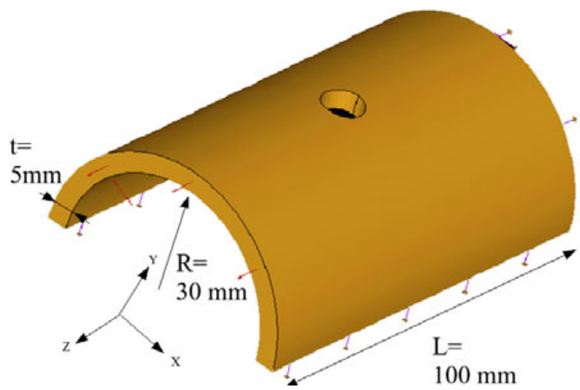

Fig. 8

The cylinder is made of steel (Young modulus = 210 GPa and Poisson ratio = 0.3). The inner pressure is equal to 10 MPa. The radius of the hole is 5 mm.

An axial stress is added according to the relation $\sigma_z = \dfrac{Pr}{2t}$ on the front face (z = 0) of the model. To avoid rigid body motions, the rear face (z = -100) is fixed along the z-direction, the bottom faces are fixed along the y-direction. One of the nodes located of the intersection between the symmetry plane and the rear face is fixed along the x-direction.

Fig. 9 shows the circular crack inserted in the symmetry plane.

- Radius = 2.5 mm.
- x = 0, y = 27.5 mm, z = -45mm.

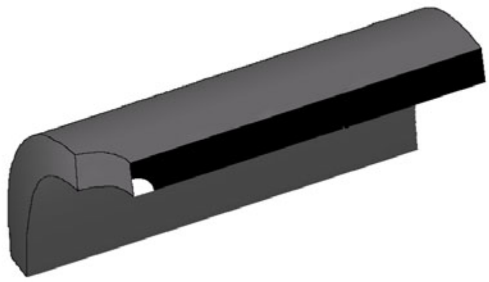

Fig. 9

Fig. 10 shows the stress intensity factor computed along the crack front. As the crack is in a symmetry plane, only the mode I is present. The highest stress intensity factor is obtained at the extremities of the crack and the lowest value is obtained at 45° with reference to the z-direction.

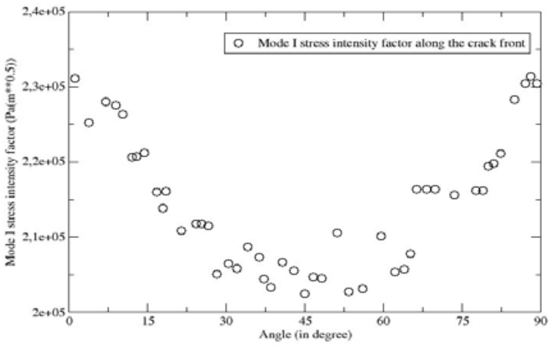

Fig. 10

Coupling FE shell with XFE plane stress formulation

A large number of shell and plates formulations have been developed and used in order to simulate the response of large structures (e.g.: outer skin of fuselage, leading edge…).

In this context, the study of crack propagation in shell is an important but far more complex problem.

Dolbow and co-workers [10] were the first to propose an implementation of the XFEM for a Mindlin Reissner plate. Nevertheless, it is impractical to re-write a XFE shell formulation for every kind of shell in all FE-software. This is the reason why we have decided to investigate the multi-level approach.

In the application presented here, a local plane stress formulation is defined on the cracked subdomain under the assumption that out-of-plane loading can be neglected.

Fig. 11 shows the sample used to validate our concept.

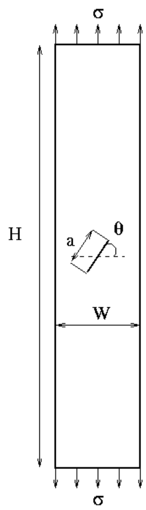

Fig.11

where $H/W = 2.5$. A local plane stress formulation is used on the cracked domain defined around the crack.

Two crack configurations have been tested and compared with analytical values:

- $a/W = 0.2$ and $\theta = 0°$,
- $a/W = 0.1$ and $\theta = 45°$.

The results (summarized in table 1) show a good agreement with the analytical values.

The following step in our development strategy is the extension to a 3D “cracked subdomain” in this way, out-of-plane effects and non-planar crack propagation are taken into account.

Table 1

Configuration	Computed (Pa $m^{0.5}$)	Analytical (Pa $m^{0.5}$)
$a/W = 0.2$ $\theta = 0°$	$K_I = 8.207$ $K_{II} = 2.64\ 10^{-6}$	$K_I = 8.122$ $K_{II} = 0$
$a/W = 0.1$ $\theta = 45°$	$K_I = 2.816$ $K_{II} = 2.876$	$K_I = 2.819$ $K_{II} = 2.819$

Conclusions and prospects

The X-FE and the Level Set methods are promising methods in modelling fracture mechanics problems. Nevertheless, these methods are not yet available in commercial software due to some inherent theoretical and implementational difficulties of the methods.

In this context, two substructuring FE-XFE based methods were developed: the super-element method and the FETI method. These methods give good results except for the overestimation of the stress intensity factor K_{II} due to spurious stress concentration for the problem of inclined crack. Those peaks are related to basic issues related to the X-FEM.

Compared with the super-element, FETI is cheaper in terms of required memory and also faster in the case of problems with more than 100000 dof's. However, if for a given crack configuration, multiple load cases must be treated, the super-element (computed only once) becomes competitive with FETI.

The two applications presented here show that the FE-XFE-FETI method can be easily used for treating industrial problems with the extended finite element method.

In the near future, the coupled FE shell / XFE formulation will be extended for 3D "cracked domains". The extension to 3D will allow for the computation of shells submitted to bending and shear stress.

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