On optimal configurations in supersonic flow

V.I. Lapygin, T.V. Sazonova*, G.E.Yakunina***
**Central Research Institute of Machine Building (TSNIIMASH)*
4, Pionerskay Str., Korolev, Moscow region, 141070, Russia
***State University of Management*
99, Ryazansky Prosp., Moscow, 109542, Russia

Abstract
Analytical solutions for variational problems on configurations of three-dimensional bodies with maximal lift-to-drag ratio at given base area or planform area are found within the limits of localised interaction between the supersonic flow and body surface. Functionals of considered variational problems depend on derivatives of desired function with respect to independent variables only, and this simplifies the solution and allows studying the structure of extremal surface. It is shown that the lower surface of optimal bodies is planar. If base area is given, the upper surface is cylindrical with generating line parallel to oncoming flow velocity vector. If planform area is given, the optimal body is a flat plate with the highest possible value of lift-to-drag ratio at prescribed Mach number and friction coefficient. The optimal body with planar upper surface is a wedge. These results are valid if the base pressure is taken into account and also for zero base pressure.

1. Introduction

Solutions of variational problems on aerodynamics of three-dimensional configurations are based on simplified models of supersonic flow interaction with a body. The local methods with the pressure coefficient defined by the angle between a normal to the body surface and the velocity vector of oncoming flow are examples of such models. Using the local methods there were found analytical solutions and examined configurations of planar, axisymmetric, and three-dimensional bodies of minimal drag [1] and, within the limits of slender bodies, of maximal lift-to-drag ratio [2-5].

Construction of a solution of variational problem about configuration of three-dimensional body is connected with integration of partial differential equations with unknown function of two variables. However, sometimes the functionals for certain variational problems on optimal configuration have the following form:

$$\Phi = \iint_S F(u, w)dxdz$$

$$y + f(x,z) = 0\,,\ u = \partial f / \partial x\,,\ w = \partial f / \partial y\,.$$

Extremals of the functional Φ are defined by a system:

$$\frac{\partial F}{\partial u} = 0\,,\ \frac{\partial F}{\partial w} = 0\,,$$

which solutions are u_i = const and w_i = const. In other words, the extremals are the planar surfaces

$$y + u_i x + w_i z + c_i = 0\,.$$

In particular, considering the problem about the configuration of minimal drag Cx_{min}, it is easy to see that with given base area the functional to be optimised has a form:

$$\Phi = \iint_{\Sigma} F(\alpha) dydz \ ,$$

$\alpha = (1 + u^2 + w^2)^{-1/2}$, and the solutions are surfaces satisfying the condition

$$\alpha = \alpha^* = \text{const},$$

where α^* corresponds to minimum of the function $F(\alpha)$ at a segment [0,1].
This condition is satisfied by circular cone surfaces with semi-vertex angle $\beta^* = \arcsin(\alpha^*)$ and planes, tangent to this cone. Infinite set of optimal bodies with single value Cx_{min} could be constructed combining the segments of these surfaces [6-8]. Examples of optimal three-dimensional bodies are illustrated in Fig.1.

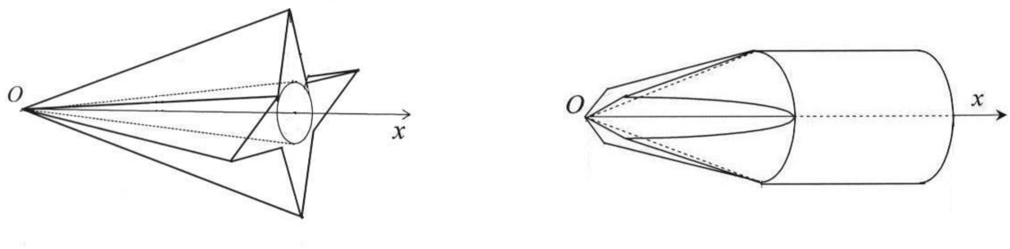


Figure 1

2. Optimal configuration of given base shape

2.1 Problem statement

Suppose that the velocity vector $\bar{v}$ of oncoming supersonic flow is parallel to X-axis of body-axis Cartesian right system of coordinates OXYZ, $\bar{v} = -\bar{x}$, where $\bar{x}$ – unit vector of X-axis. Y-axis is pointing upwards. Let's consider bodies with planar base situated in YOZ plane and the condition $\alpha = (\bar{n}, \bar{v}) > 0$ is satisfied on the surface $x = f(y,z)$. Here $\bar{n}$ – unit vector of internal normal to a body surface element, and pressure coefficient C_p depends on α and Mach number M. Suppose also that friction coefficient C_f is constant on the body surface, and tangential stress vector lies in the plane of vectors $\bar{n}$ and $\bar{v}$.

Lift and drag aerodynamic coefficients are written in form:

$$S_b C_y = \iint [C_p(\alpha) - C_f \alpha/g]\, u dydz \tag{1}$$

$$S_b C_x = \iint [C_p(\alpha) + C_f g/\alpha]\, dydz \tag{2}$$

$$S_b = \iint dydz$$
$$\alpha = (1+u^2+q^2)^{-1/2},\ u = \partial f/\partial y,\ w = \partial f/\partial z,\ g = (1-\alpha^2)^{1/2}$$

Integration is taken over the body base area S_b.
Lift-to-drag ratio is defined by the formula

$$K = C_y / C_x \tag{3}$$

and the problem is to find a function $f(y,z)$, which realises maximum of functional (3) with given area S_b. The first variation of the functional (3) is $\delta K = (\delta C_y - K\delta C_x)/C_x$, therefore it follows from the condition $\delta K = 0$ that the problem is to find extremum of the functional

$$\Phi = \iint \mathrm{F}(\alpha, u) dydz$$
$$\mathrm{F}(\alpha, u) = [C_p(\alpha) - C_f \alpha/g]\, u - \mathrm{K}\, [C_p(\alpha) + C_f g/\alpha] + \lambda.$$

Here λ – constant Lagrangian coefficient, and α, g, u – functions of variables y and z.

2.2 Analysis of extremal surfaces

Equations of extremal surfaces are defined by the system:

$$\frac{\partial F}{\partial u} = 0\,,\ \ \frac{\partial F}{\partial w} = 0\,,$$

which has two families of solutions [9]:

$$w = 0,\quad C_p(\alpha) - \alpha g^2 C_p'(\alpha) - K[C_f - \alpha^2 g\, C_p'(\alpha)]\mathrm{sign}(u) = 0 \tag{4}$$

$$u[C_p'(\alpha) - C_f/g^3] - K[C_p'(\alpha) - C_f/(\alpha^2 g)] = 0,\quad C_p(\alpha) = C_f\alpha/g \tag{5}$$

$$C_p'(\alpha) = dC_p(\alpha)/d\alpha.$$

The solution (4) defines a set of planes parallel to Z-axis: $x + u_1 y + c_1 = 0,\ c_1 = \mathrm{const.}$
The solution (5) defines a set of planes (symmetrical respectively to the plane XOY) that do not create lift:

$$x + u_2 y + w_2 y + c_2 = 0,\ \ w_2 = \pm[1 - \alpha_2^2(1 + u_2^2)]^{1/2} / \alpha_2\,,\ \ c_2 = \mathrm{const.}$$

Optimal body should be formed by segments of the planes (4) and (5) so that the lower surface is parallel to Z-axis and is situated at the angle of attack $\beta_1 = \arcsin(\alpha_1)$ to the flow. The upper surface is formed by two symmetrical planes (5).

Analysis of deduced extremal surface has shown that K maximum is realised at $\alpha_2 \to 0$, and formulation of the problem should includes the surfaces with $\alpha = 0$ [9].

2.3 Construction of a solution

Lift-to-drag ratio is determined by the expression

$$K = C_y / (C_x + C_f \Delta_0),\ \ \Delta_0 = S_0 / S_b\,, \tag{6}$$

where C_y, C_x – lift and drag coefficients of a body with $\alpha > 0$,
S_0 – area of body surface with $\alpha = 0$.
The condition of the functional (6) extremum is that the first variation vanishes:

$$\delta K = \delta(C_y - KC_x) - KC_f\delta\Delta_0 = 0 \tag{7}$$

It follows that a section of body surface with $\alpha > 0$ is the plane (4).

The section with $\alpha = 0$ is cylindrical surface, which generating line equation is the functional Δ_0 extremal.
Let the body span is $l = 2z_k$. The upper body surface with α=0 may consist of three sections: one curvilinear section and two planar sections situated symmetrically respectively to YOX plane and parallel to it.

Area of curvilinear section is determined by integral:

$$S_{01} = 2u_1 \int_0^{z_k} y(1 + y'^2)^{1/2}\, dz$$

$y' = dy/dz$, $y(z)$ – projection of curvilinear section at YOZ plane.

Area of the sections parallel to YOX:

$$S_{02} = u_1 y_k^2\,,\ \ y_k = y(z_k)\,.$$

Total area of the upper surface

$$S_0 = S_{01} + S_{02} = u_1\delta_0\,, \quad \delta_0 = 2\left[\int_0^{z_k} y(1+y'^2)^{1/2}\,dz + y_k^2/2\right].$$

Therefore the problem is to find minimum of the functional

$$\Phi = y_k^2/2 + \int_0^{z_k} y[(1+y'^2)^{1/2} + \lambda_1]dz\,, \tag{8}$$

λ_1 – constant Lagrangian coefficient.

It follows from the analysis of the functional (8) extremals that if the coordinate z_k is not given then optimal body is two-dimensional wedge with upper surface parallel to the velocity vector $\overline{v}$.

If the body thickness $y(0) = y_0$ and area S_b are given

$$y(z) = -tz + y_0\,,\ t = y_0^2/S_b$$

then optimal body is delta wing with wedge-like profile.

If the body span $l = 2z_k$ and base area are given then the base contour is determined by the relations:

$$\begin{aligned}
&z = [\,\lambda_1(ay^2 + by + c)^{1/2} + c_1 I\,]/a,\\
&a = 1 - \lambda_1^2,\ \ b = 2\lambda_1 a/(1 + I^*),\ \ c = -\,a^2/(1 + I^*),\ \ c_1 = -\,c/a,\\
&I^* = |\,a\,|^{-1/2}\,[\pi/2 + \arcsin(|\,\lambda_1|^{-1})],\\
&I = |\,a\,|^{-1/2}\,[\pi/2 + \arcsin(|\,\lambda_1| - (1 + I^*)y)]
\end{aligned} \tag{9}$$

Here all linear dimensions are given in ratio to z_k.

Optimal contours of base area are presented in Fig.2 with different $\Delta_k = S_b/2z_k^2$; three-dimensional view of optimal body is illustrated in Fig.3. The body base shape does not depend on u_1, M, C_f, and, consequently, on specific form of the dependence $C_p(\alpha)$.

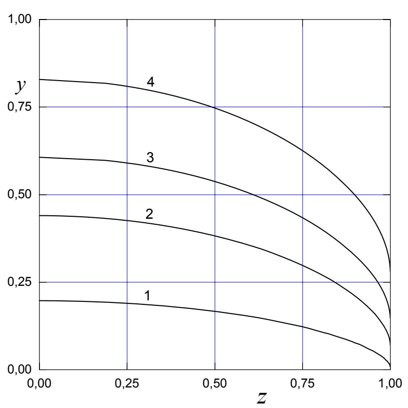


Figure 2

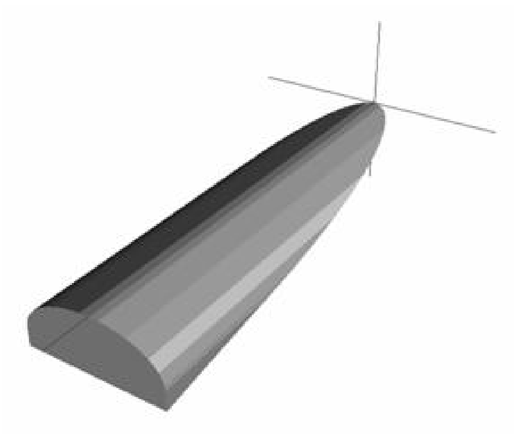


Figure 3

Let's determine the angle of attack β_1 and maximal lift-to-drag ratio at γ=1.4, using the local wedge formula [8]

$$C_p(\alpha) = \frac{\gamma+1}{2}\left[1 + \left(1 + \left(\frac{4}{(\gamma+1)\alpha\sqrt{M^2-1}}\right)^2\right)^{1/2}\right]. \tag{10}$$

The values β_l^* and K^* for optimal two-dimensional wedge at γ=1.4 are presented in Tables 1 and 2, correspondently (the upper figures in the cells).

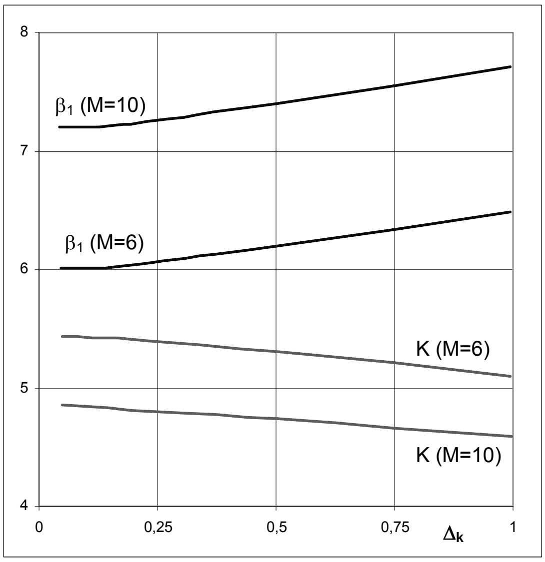
Table 1

M	β_1^*, degree		
	C_f = 0.001	0.002	0.003
6	4.32 4.41	6.00 6.12	7.24 7.51
10	5.32 5.52	7.19 7.39	8.49 8.71
15	5.96 6.10	7.82 8.01	9.11 9.31
∞	6.75 7.00	8.50 8.70	9.72 10.02

Table 2

M	K^*		
	C_f = 0.001	0.002	0.003
6	7.34 7.50	5.42 5.53	4.55 4.60
10	6.37 6.50	4.83 5.00	4.13 4.40
15	5.96 6.20	4.61 4.81	3.97 4.20
∞	5.59 5.80	4.41 4.61	3.84 4.05

For bodies defined by the formulas (9) the optimal angle β_1 and maximal lift-to-drag ratio K are plotted against Δ_k in Fig.4 for C_f = 0.002 and M=6; 10. Note that β_1 values are close to β_1^* ($\beta_1 - \beta_1^* < 30'$).



$$\Delta_k = \frac{S_b}{2z_k^2}\int_{y_0}^{y_k} z\,dy + y_k$$

Figure 4

Values of lift-to-drag ratio of optimal bodies with triangular and curvilinear base with similar area S_b and length L are close. At the same time span of optimal delta wing is 1.5 times greater than span of a body defined by the relations (9).

If the base contour is given, the curve $y = y(z)$ is guiding line of the upper cylindrical surface with generating lines parallel to X-axis. Configurations of optimal bodies with various base shape are illustrated in Fig.5 at $S_b/y_o^2 = \pi/2$, M = 15, C_f = 0.002. Corresponding pairs of (β_1, K) values are: *1* – (8.21°, 4.40), *2* – (8.09°, 4.46), *3* – (8.1°, 4.45).

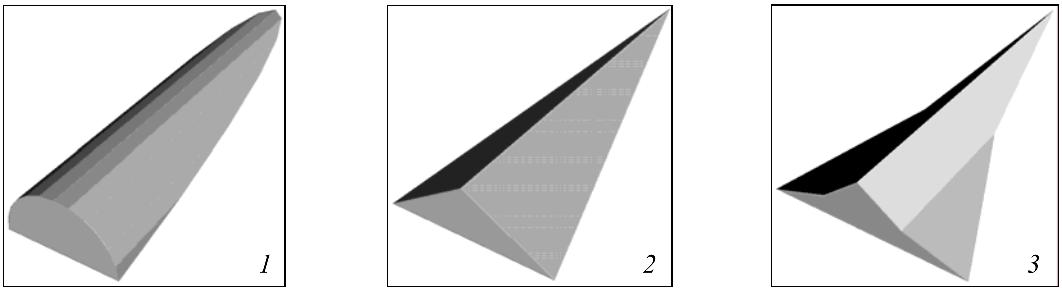


Figure 5

3. Optimal configuration of given planform

Let's consider bodies, such that depression flow is realised at a part of their surface. The equation for the lower body surface takes form: $y + f_1(x, z) = 0$, and for the upper surface: $y + f_2(x, z) = 0$. Here at the lower surface the pressure coefficient $C_{p1} > 0$, and at the upper surface $C_{p2} \le 0$. With assumptions accepted in previous section, aerodynamic coefficients are determined by the formulas

$$
\begin{aligned}
&S_b C_y = \iint (C_{p1} - C_{p2} - C_f / \Delta_{11} - C_f / \Delta_{21}) dxdz \\
&S_b C_x = \iint (C_{p1} u_1 - C_{p2} u_2 + C_f \Delta_{11} + C_f \Delta_{21}) dxdz \\
&K = C_y / C_x, \ \int S_b = \iint dxdz
\end{aligned}
\tag{11}
$$

Integration is taken over a section in XOZ plane corresponding to body planform area.

$$
u_i = \frac{\partial f_i}{\partial x}, w_i = \frac{\partial f_i}{\partial z}, \Delta_i^2 = 1 + u_i^2 + w_i^2, \Delta_{i1}^2 = 1 + w_i^2, (\bar{n} \cdot \bar{v}) = \frac{u_i}{\Delta_i}, C_{pi} = C_p\left(\frac{u_i}{\Delta_i}\right), i = 1,2
$$

Determination of a configuration of maximum lift-to-drag ratio is reduced to finding the minimum of the functional

$$
\Phi = \iint F(u_i, w_i) dxdz \tag{12}
$$

$$
F = C_{p1} - C_{p2} - C_f / \Delta_{11} - C_f / \Delta_{21} - K(C_{p1} - C_{p2} + C_f \Delta_{11} + C_f \Delta_{21}) + \lambda_2,
$$

λ_2 – constant Lagrangian coefficient.

Extremal surfaces are defined by the system of equations:

$$
\frac{\partial F}{\partial u_i} = 0, \ \frac{\partial F}{\partial w_i} = 0, \ i = 1,2. \tag{13}
$$

For the upper and lower surfaces these equations have two families of solutions for each surface. The first family for the lower surface defines planar surfaces parallel to Z-axis:

$$
\begin{aligned}
&w_1 = 0 \\
&C'_{p1} \frac{1}{\Delta_1^3} - K\left(C'_{p1} \frac{1}{\Delta_1^3} + C_{p1}\right) = 0
\end{aligned}
\tag{14}
$$

The second family is defined by the equations:

$$
\begin{aligned}
&C_{p1} u_1 + C_f \Delta_{11} = 0 \\
&C'_{p1} \frac{\Delta_{11}^2}{\Delta_1^3} - K\left(C'_p \frac{\Delta_{11} u_1}{\Delta_1^3} + C_{p1} + C_{p1} \frac{u_1^2}{\Delta_1^2} + C_f \frac{\Delta_{11} u_1}{\Delta_1^2}\right) = 0
\end{aligned}
\tag{15}
$$

The first equation in (15) conflicts with the condition $C_{p1} > 0$ and corresponds to zero drag of the lower surface that has no physical meaning. Consequently, the lower surface of the optimal body is a plane parallel to Z-axis. The angle of attack for this plane $\beta_1 = \mathrm{arctg} u_1$ is determined from the equation (14_2). It can be shown by analogy that the upper body surface is also a surface parallel to Z-axis ($w_2 = 0$).

Therefore the upper and lower surfaces of the body are the planar surfaces, and the optimal body is a flat plate with the angle of attack β_1.

In fact, analysed problem statement does not include planes parallel to Y-axis, which add non-zero thickness to the optimal body. But such planes make the drag greater and do not influence upon the lift, so they reduce lift-to-drag ratio *K*. That is why such planes can't be the extremals.

Let's analyse two limiting cases of discussed problem solutions: $\beta_1(M^2-1)^{1/2} << 1$ and $\sin\beta_1(M^2-1)^{1/2} >> 1$.

For the first case:

$$C_{p1} = -C_{p2} = \frac{2\beta_1}{\sqrt{M^2-1}}\,,\ \ \beta_1 = \beta_2\,,$$

For the second case, according to (10)

$$C_{p1} = (\gamma+1)\frac{u_1^2}{\Delta_1^2}\,,\ \ C_{p2} = 0\,.$$

The equation (14_2) for the upper surface becomes an identical relation valid with arbitrary $u_2 > 0$. Therefore in this case $u_1 = u_2$ too. Correctness of the condition $u_1=u_2$ for finite values of Mach number was checked by numerical optimization of bodies of various planform with the help of the code [10]. Optimal values for β_1 and *K* of a flat plate are presented in Tables 1 and 2, correspondingly (the lower figures in cells); it is seen that at moderate Mach numbers the flat plate has lesser angles of attack β_1 and greater *K* values as compared with two-dimensional wedge.

3.1 Optimal configuration of given planform and planar upper surface

Let's examine a body, which upper surface coincides with XOZ plane, and its length equals 1. In accordance with the above analysis the lower surface is planar. Obviously, for constructing the closed body surface it is necessary to introduce a cylindrical surface with generating lines parallel to Y-axis. For this purpose consider a body formed by two planes $y = 0$ and $y = u_1x$ and cylindrical surface $z = f(x)$. The problem to find a configuration of maximal lift-to-drag ratio with known u_1 and given planform area is reduced to searching an extremal of the functional

$$\Phi_1 = \int F(x,z,z')dx\,,$$

$$F = \left(\frac{2z'^3}{1+z'^2} + C_f\sqrt{1+z'^2}\right)x + \lambda z\,.$$

Equations for the extremal and boundary conditions are written in form:

$$\lambda + \frac{d}{dx}F_{z'} = 0, F_{z'} = 0 \text{ at } x = 0 \text{ and at } x = 1\,.$$

It follows that $\lambda=0$ and the extremal is a straight line $z = c = S_b/2$.

Thus if the upper surface is planar then optimal body is a wedge. Obviously, the greater is span the greater is *K* value. It is noted that for all examined cases this solution is true also if base drag is taken into consideration on the condition that pressure coefficient in the base region is constant C_{pb}=const.

4. Conclusion

The result of the analysis with assumption about local character of force interaction between the flow and body surface are analytical solutions on configurations of maximal lift-to-drag ratio at supersonic flow velocities. For given base area or given planform area the lower (windward) surface of the optimal body is flat. This conclusion agrees with the research results on the influence of V-shape upon a value of lift-to-drag ratio of V-shaped wings at supersonic Mach numbers [11].

Construction of solutions for variational problems was generated with minimal constraints on a body shape – either base area or planform area. Therefore values of lift-to-drag ratio of optimal configurations are extremely accessible or upper bounds of lift-to-drag ratio at supersonic velocities.

The analytical results do not conflict with numerical data on investigation of configurations like waveriders [12, 13]. This is additional argumentation in favour of taken local approach, which allows, the same as for the problem on a body of minimal drag [6-8], finding the solutions consistent with the solutions found with more accurate assumptions about the character of force interaction between a flow and body surface.

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