An Improved Shape-Based Method for Interplanetary Low-Thrust Orbit Design

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Abstract

As low-thrust thrusters were applied in Deep Space 1 spacecraft successfully, interplanetary low-thrust orbit design and optimization is becoming increasingly popular. However, orbit design and optimization is a very challenging and time-consuming task. So a suitable preliminary orbit, which is beneficial for optimizers to converge to a more accurate trajectory quickly, is extremely important. The shape-based method is an effective preliminary orbit design method. However, the traditional shape-based methods without considering the first order optimal necessary conditions cannot guarantee an optimal preliminary orbit, and the thrust direction is constrained to be tangential. In this paper, an improved shape-based method using Fourier series is proposed, which can avoid these shortages above mentioned easily. Firstly, the spacecraft dynamics model under the polar coordinate is given. Secondly, the first order necessary conditions are derived from the Hamilton function, and the optimal control problem is converted to a nonlinear programing problem about Fourier series coefficients. Thirdly, Matlab *fmincon* function is used to solve the nonlinear programing problem. Lastly, a simple Earth-Mars rendezvous is used to depict the proposed method's feasibility in interplanetary low-thrust orbit design, and some comparison results with traditional shape-based methods are utilized to prove the advantages of the proposed method in offering initial guess for optimizers.

1. Introduction

In 1998, electric propulsion thrusters were applied in NASA's Deep Space 1 mission successfully, which verified the potential value of the continuous low-thrust system. Recently, interplanetary continuous low-thrust orbit design and optimization is becoming increasingly popular. However, it is a very challenging and time-consuming task.

Particularly, continuous low-thrust orbit design consists of two phases: preliminary design phase and precise design phase. To pursuit a faster optimization speed and a more accurate trajectory, preliminary design phase is expected to provide an efficient initial trajectory guess for optimizers. Shape-based (SB) method is one of the most efficient preliminary methods. In SB methods, some functions are assumed to present the spacecraft trajectory, and then boundary conditions are used to compute the function parameters, finally the needed thrust during motion can be got analytically.

Now many kinds of SB methods have been proposed by researchers. For instance, Petropoulos et al. developed an exponential sinusoid (ES) method for two dimensional interplanetary transfer orbit design problem in reference[1-2]. The ES method constrains the thrust direction to be tangential, which has a function form given in Eq. (1):

$$r = k_0 e^{k_1 \sin(k_2\theta+\phi)} \tag{1}$$

where k_0, k_1, k_2 and ϕ are constants. Izzo[3] utilized this method to investigate the multi-revolution Lambert's problem, and simplified the planet and low-thrust trajectory design procedure. Cui and his partners[4] proposed a new search approach for launch window of low-thrust gravity-assist mission based on ES method, and the proposed algorithm has fewer searching variables and more efficient compared with traditional SB algorithms. But the ES method cannot satisfy the circle terminal orbits unless thrusters provide impulse propulsion; besides convergent shape parameters cannot be solved as other constraints are introduced.

Zheng et al. [5] proposed a new trajectory shape called logarithmic spiral (LS) method, which has a function form given in Eq. (2):

$$r = r_0 e^{q(\theta-\theta_0)} \tag{2}$$

where r_0 is initial geocentric distance, θ_0 is initial phase angle, and q is a constant that need to design. The feasibility and essential characteristics of the logarithmic spiral orbit are analyzed. And analytical geocentric distance r expression and phase angle θ expression about flight time subject to a tangential thrust are derived. But the LS method cannot satisfy the terminal constraints as same as the ES method.

To overcome these disadvantages mentioned above, Wall et al.[6-7] developed a two-dimensional, seven-parameter inverse polynomial (IP) method, which is assumed to be a form given in Eq. (3):

$$r = \frac{1}{a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5 + g\theta^6} \tag{3}$$

where (a,b,c,d,e,f,g) are shape parameters solved through initial and terminal conditions. But the thrust direction is also constrained to be tangential in the IP method, and it cannot handle the thrust constraint very well. Shang[8] proposed a semi-analytical Lambert algorithm based on the N-degree IP approach in order to improve the precision of primary design for an interplanetary low-thrust transfer trajectory.

Ehsan and Abdelkhalik[9-10] proposed a new shape-based trajectory design method using Fourier series. With a hypothesis of tangential thrust, a preliminary orbit that satisfy maximum thrust constraint is designed by this SB method.

In a word, all recent SB methods design spacecraft trajectory based tangential thrust assumption, and cannot guarantee the designed trajectory is optimal one without the first order optimal necessary conditions. So an improved shape-based method using Fourier series is proposed in this paper, which can avoid these shortages above mentioned easily. Firstly, the spacecraft dynamics model under the polar coordinate is given. Then, the first order necessary conditions are derived from the Hamilton function, and the optimal control problem is converted to a nonlinear programing problem about Fourier series coefficients. Lastly, a simple Earth-Mars rendezvous is used to depict the proposed method's feasibility in interplanetary low-thrust orbit design, and some comparison results with traditional shape-based methods are utilized to prove the advantages of the proposed method in offering initial guess for optimizers.

2. Spacecraft orbital model

In the polar coordinate, the spacecraft orbital motion without considering any perturbation and celestial bodies' rotation can be written as Eq. (4)

$$\begin{cases} \dot{r} = u \\ \dot{\theta} = v/r \\ \dot{u} = -\mu/r^2 + v^2/r + a\sin\alpha \\ \dot{v} = -uv/r + a\cos\alpha \end{cases} \tag{4}$$

where, as shown in Fig. 1, r is the magnitude of the position vector $\boldsymbol{r}$, θ is the polar angle, u is the magnitude of spacecraft radial velocity vector $\boldsymbol{u}$, v is the magnitude of spacecraft circumferential velocity vector $\boldsymbol{v}$, μ is the gravitational parameter, a is the spacecraft thrust-acceleration magnitude, α is the steering angle, and ξ is the flight path angle.

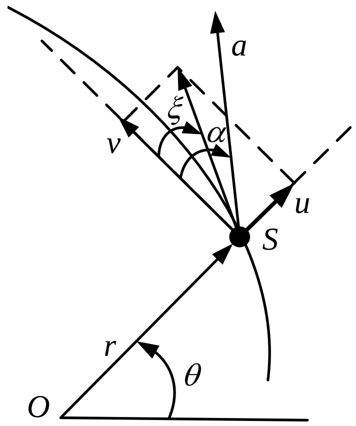

Figure 1 Trajectory variables

3. Shape-based method using Fourier series

3.1 Index performance and boundary conditions

Usually the index performance in interplanetary low-thrust orbit design is set to minimize the flight time or fuel consumption. In this paper, we choose minimum characteristic velocity (i.e. minimum fuel consumption), as shown in Eq. (5)

$$J = \min \Delta V = \int_{t_i}^{t_f} a \, dt \tag{5}$$

To accomplish spacecraft manoeuver mission successfully, some constraints, as shown in Eq. (6), should be reached.

$$P\big|_{i,f} = P_{i,f} \; , V\big|_{i,f} = V_{i,f} \; , A\big|_{i,f} = A_{i,f} \tag{6}$$

where (P, V, A) represent spacecraft's position, velocity and acceleration conditions respectively; (i, f) represent initial and terminal moment respectively.

3.2 Traditional Fourier series (TFS) method

In a fixed-time problem, as shown in Fig. 1, it is assumed that the thrust is aligned along or against the velocity vector, i.e. $\alpha = \xi + n\pi$, where $n = 0, 1$.

From the fourth equation of Eq. (4), one can write

$$a = \frac{r\dot{v} + uv}{r\cos\alpha} \tag{7}$$

Substituting the Eq. (7) into the third equation of Eq. (4), one can write

$$\dot{u} + \frac{\mu}{r^2} - \frac{v^2}{r} = \frac{r\dot{v} + uv}{r}\tan\alpha \tag{8}$$

where the tangential thrust assumption can be written as

$$\tan\alpha = \tan\xi = \frac{u}{v} = \frac{\dot{r}}{r\dot{\theta}} \tag{9}$$

Substituting the first and second equations of Eq. (4) and the tangential thrust assumption into Eq. (8), one can be rewritten as

$$r^2\left(\ddot{r}\dot{\theta} - \dot{r}\ddot{\theta}\right) + \dot{\theta}\left(\mu - 2r\dot{r}^2\right) - \left(r\dot{\theta}\right)^3 = 0 \tag{10}$$

According to Fourier series theory, the radius r and the polar angle θ can be approximated as follows:

$$x = \frac{a_{x0}}{2} + \sum_{i=1}^{n_x}\left\{a_{xi}\cos\left(\frac{n\pi t}{T}\right) + b_{xi}\sin\left(\frac{n\pi t}{T}\right)\right\} \tag{11}$$

where x means (r, θ), n_x is the number of Fourier terms, (a_{x0}, a_{xi}, b_{xi}) are Fourier coefficients, T is the total flight time.

Substituting the state approximations Eq. (11) into Eq. (10), the differential is converted to a nonlinear algebraic equation, in which the only unknowns are the Fourier coefficients and the independent time variable:

$$F\left(a_{x0}, a_{xi}, b_{xi}; t\right) = 0 \tag{12}$$

It can be found that Eq. (5), Eq. (6) and Eq. (12) result a nonlinear programming problem about Fourier coefficients.

3.3 Improved Fourier series (IFS) method

In terms of Eq. (4) and Eq. (5), the Hamilton function is written as

$$H = a + \lambda_r u + \lambda_\theta \frac{v}{r} + \lambda_u\left(-\frac{\mu}{r^2} + \frac{v^2}{r} + a\sin\alpha\right) + \lambda_v\left(-\frac{uv}{r} + a\cos\alpha\right) \tag{13}$$

Based on the optimal control theory, co-state equations and control equations are shown as

$$\begin{cases} \dfrac{\mathrm{d}\lambda_r}{\mathrm{d}t} = -\dfrac{\partial H}{\partial r} = \lambda_\theta \dfrac{v}{r^2} - \lambda_u \dfrac{2\mu}{r^3} + \lambda_u \dfrac{v^2}{r^2} - \lambda_v \dfrac{uv}{r^2} \\ \dfrac{\mathrm{d}\lambda_\theta}{\mathrm{d}t} = -\dfrac{\partial H}{\partial \theta} = 0 \\ \dfrac{\mathrm{d}\lambda_u}{\mathrm{d}t} = -\dfrac{\partial H}{\partial u} = -\lambda_r + \lambda_v \dfrac{v}{r} \\ \dfrac{\mathrm{d}\lambda_v}{\mathrm{d}t} = -\dfrac{\partial H}{\partial v} = -\lambda_\theta \dfrac{1}{r} - \lambda_u \dfrac{2v}{r} + \lambda_v \dfrac{u}{r} \end{cases} \tag{14}$$

$$\begin{cases} \dfrac{\partial H}{\partial a} = 1 + \lambda_u \sin\alpha + \lambda_v \cos\alpha = 0 \\ \dfrac{\partial H}{\partial \alpha} = \lambda_u a \cos\alpha - \lambda_v a \sin\alpha = 0 \end{cases} \tag{15}$$

With $X = \dot{u} + \mu/r^2 - v^2/r$ and $Y = \dot{v} + uv/r$, the third and fourth equation can be rewritten as

$$\begin{cases} X = a\sin\alpha \\ Y = a\cos\alpha \end{cases} \tag{16}$$

From Eq. (16), the spacecraft thrust-acceleration magnitude can be solved

$$a = \sqrt{X^2 + Y^2} \tag{17}$$

Meanwhile, the thrust-acceleration direction could be defined by the results of $\sin\alpha$ and $\cos\alpha$.

From Eq. (15) and Eq. (16), one can write

$$\begin{cases} \lambda_u X + \lambda_v Y = -a \\ \lambda_u Y - \lambda_v X = 0 \end{cases} \tag{18}$$

Solving Eq. (18), we can get the expressions of co-state variables (λ_u, λ_v)

$$\begin{cases} \lambda_u = \dfrac{-aX}{X^2 + Y^2} \\ \lambda_v = \dfrac{-aX}{X^2 + Y^2} \end{cases} \tag{19}$$

According to the second and fourth equation of Eq. (14), the value of constant λ_θ can be represented as

$$\lambda_\theta = \left(\lambda_v u - 2\lambda_u v - \dot{\lambda}_v r\right)\Big|_i \tag{20}$$

According to the first and fourth equation of Eq. (14), one can wrote

$$r^3 \dot{\lambda}_r + rv\left(\lambda_v u - \lambda_u v - \lambda_\theta\right) + 2\mu\lambda_u = 0 \tag{21}$$

where co-state variable λ_r can be solved through the third equation of Eq. (14)

$$\lambda_r = \lambda_v \frac{v}{r} - \dot{\lambda}_u \tag{22}$$

So all co-states variables $(\lambda_r, \lambda_\theta, \lambda_u, \lambda_v)$ can be expressed as function of state variables (r, θ).

Substituting the state approximations Eq. (11) into Eq. (21), the differential is converted to a nonlinear algebraic equation, in which the only unknowns are the Fourier coefficients and the independent time variable:

$$F_{opt}\left(a_{x0}, a_{xi}, b_{xi}; t\right) = 0 \tag{23}$$

It can be found that Eq. (5), Eq. (6) and Eq. (23) result a nonlinear programming problem about Fourier coefficients.

4. Spacecraft orbit design process using IFS method

The spacecraft manoeuver orbit design process using IFS method is shown in Fig. 2, in which FCs is short for ‘Fourier coefficients’.

The *fmincon* function is a MATLAB command used to solve multi-variable, nonlinear, optimal problem with constraints. According to *fmincon* function requirement, it is necessary to obtain a set of initial guess for Fourier coefficients. In reference [9], some rough approaches used to gain initial guess are proposed. A constraint about Fourier terms $n_x \geq 2$ is set to satisfy the boundary conditions. Besides, although there is no upper limit on the number of included Fourier terms, the computational efficiency and precision is an important consideration.

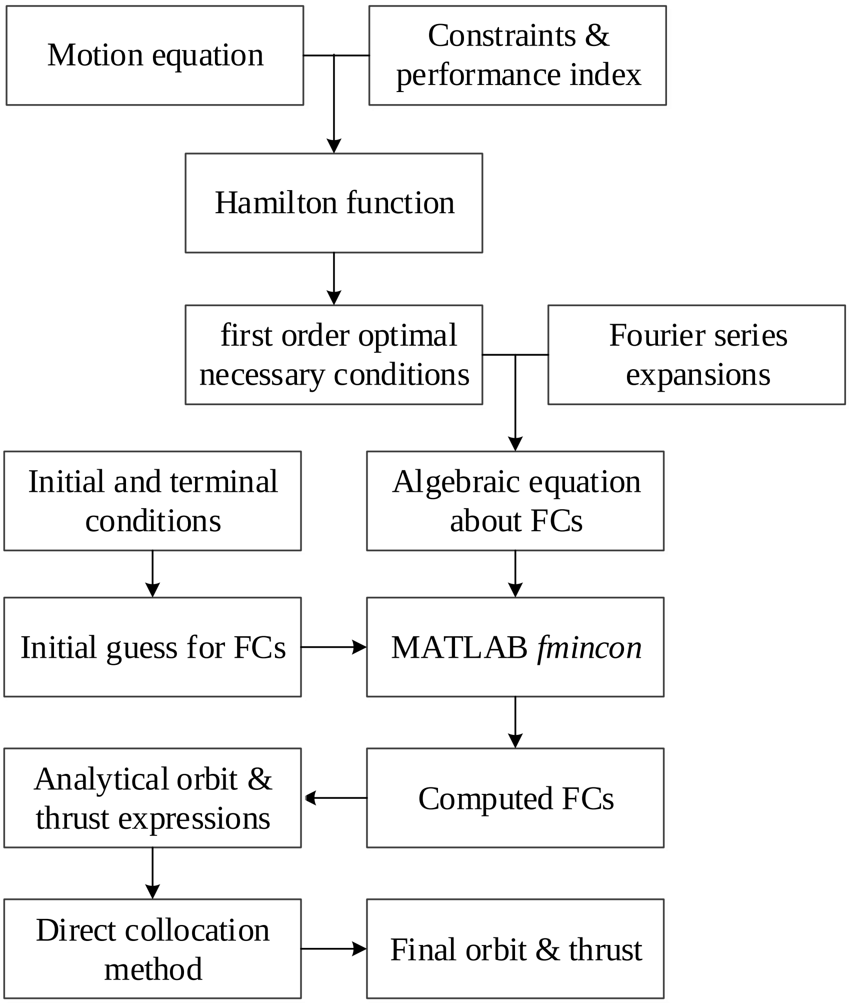

Figure 2 Flow chart of orbit design using IFS method

In process of IFS method for designing interplanetary low-thrust trajectory, a cubic polynomial function is used to get initial guess for Fourier coefficients in this paper. Based on the initial guess, new Fourier coefficients are computed using *fmincon* function. Then, analytical orbit and thrust expressions about Fourier coefficients are found. These expressions are used to offer initial guess for optimizers. After the preliminary design phase, trajectory optimization is required. Here, direct collocation method is applied to optimized spacecraft maneuver orbit. In direct collocation method, the optimal control problem is converted into a nonlinear programming problem. The flight time history is discretized into N intervals, and two endpoints of each intervals are called 'node'. Polynomial interpolants are used to approximate solutions over intervals of the total time history. These polynomials are computed using collocation point selection based on Jacobi polynomials. The resulting polynomial interpolants take on the form of a family of modified-Gaussian quadrature rules known as the Gauss-Lobatto rules [11].

5. Simulation example

5.1 Earth-Mars transfer

The test case has been performed at MATLAB 2014a on an Intel Core i5 2.6GHz with Windows 8.

The continuous low-thrust Earth-Mars transfer is considered. In this case, canonical units are used, such as 1 distance unit (DU) is 1 AU and 2π time unit (TU) is 1 year. The boundary conditions and input parameters are listed in Table 1, in which N_{rev} means the number of revolutions about the Sun and N_{nod} is the number of nodes in direct collocation method.

Fig. 3 and Fig. 4 are spacecraft's transfer orbit and thrust acceleration profile using different SB methods respectively. Fig. 5 and Fig. 6 are optimized transfer orbit and thrust acceleration profile using direct collocation method based on different initial guesses from different SB methods. Table 2 is simulation results using different initial guesses.

Table 1: Boundary conditions and input parameters for Earth-Mars transfer

Boundary conditions		Input parameters	
r_i	1 DU	N_{rev}	1
θ_i	0 rad	n_r	6
r_f	1.5234 DU	n_θ	6
θ_f	9.831 rad	T	13.447 TU
$\dot{r}_i$	0 DU/TU	N_{nod}	151
$\dot{\theta}_i$	1 rad/TU		
$\dot{r}_f$	0 DU/TU		
$\dot{\theta}_f$	0.5318 rad/TU		

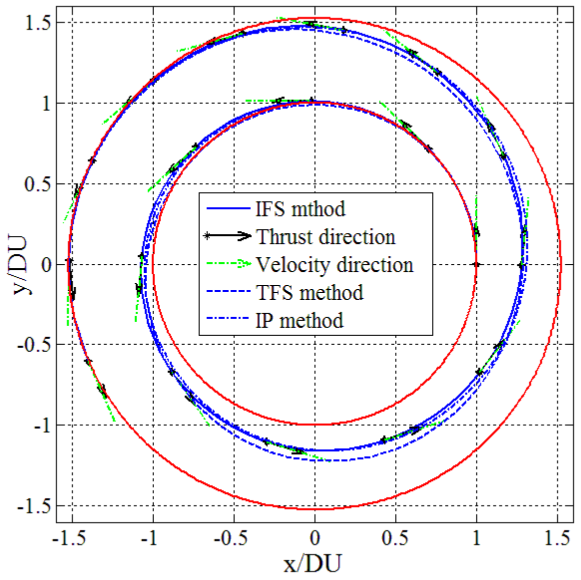

Figure 3 Transfer orbit using different SB methods

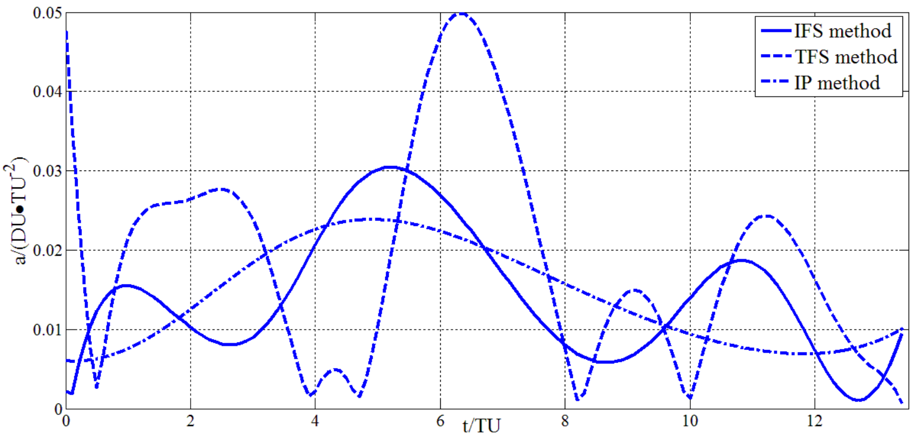

Figure 4 Thrust acceleration profile using different SB methods

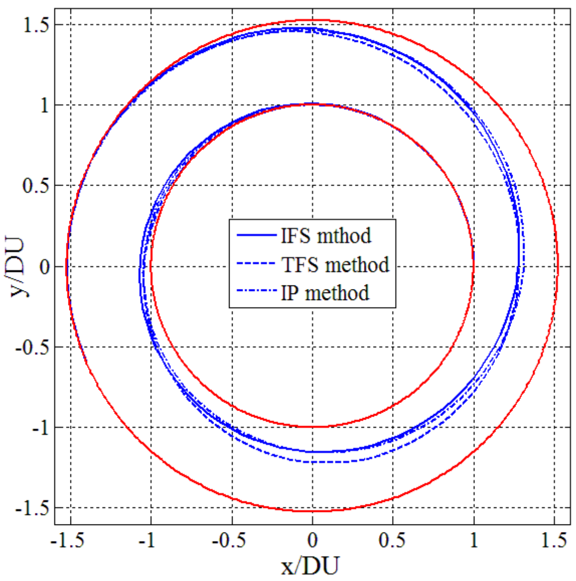

Figure 5 Optimized transfer orbit using direct collocation method

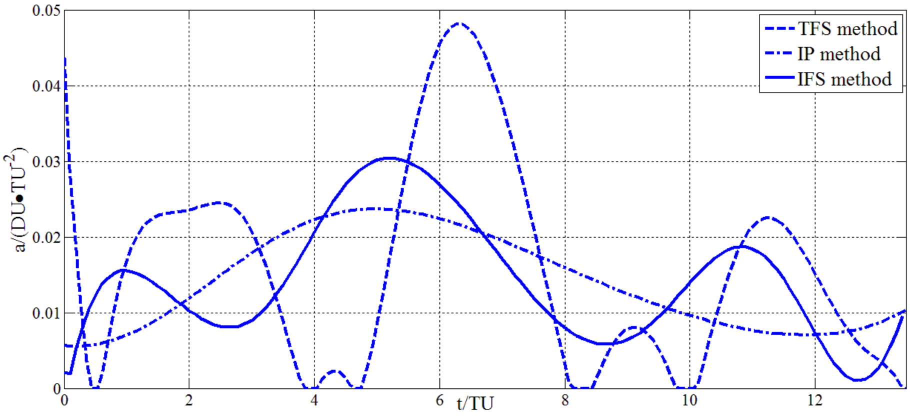

Figure 6 Optimized thrust acceleration profile using direct collocation method

Table 2 Simulation results using different initial guesses

	ΔV before optimization	ΔV after optimization
IFS method	0.1896 DU/TU	0.18787 DU/TU
TFS method	0.2514 DU/TU	0.18908 DU/TU
IP method	0.1951 DU/TU	0.18789 DU/TU
Hohmann transfer	0.1877 DU/TU	

4.2 Simulation analysis and conclusion

1) Compared with other SB methods, which have to suppose that acceleration direction is parallel with velocity direction, the IFS method do not need to constrain the relation about acceleration direction and velocity direction. As shown in Fig. 2, little difference between acceleration direction and velocity direction can be found in each points.
2) Due to less number of shape parameters, the thrust acceleration profile using IP method is smoother than others. And the thrust acceleration profile using TFS method is the roughest as shown in Fig. 3 and Fig. 5. Except TFS method, the acceleration profile before and after optimization varied unobviously, because of the unobvious decrease of characteristic velocity.
3) In Table 2, it is evident to find IFS method can provide better initial guess for optimizers. Either the preliminary design phase or the precise design phase, the IFS method can gain the minimum fuel consumption transfer orbit.

Acknowledgement

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