Helicopter autopilot based on precise nonlinear equations of the main rotor with Hiller hub

A. E. Barabanov *
Saint Petersburg State University
Universitetskij pr., 28. Saint Petersburg, 198504 Russia
e-mail: andrey.barabanov@gmail.com

Abstract

A cross influence between the blade pitch angle and the servo blade flapping angle is studied. A full mechanical equations of a rigid body motion was taken as a basis for a description of the flapping motion process. It was found that the dimensionless variables in the equations can be divided into groups by their decimal orders. The equations were truncated up to the second order of approximation error.

1. Introduction

An autopilot was designed for the small electrical helicopter Walkera 450. It has the Hiller hub, the main rotor diameter is 700 mm, the weight is 830 g, and the useful load is 300 g. The helicopter is controlled by four signals transmitted from the control panel. The first two controls determine angles of the swash plate, the other two controls determine collective pitches of the main and tail rotors. The main rotor frequency depends also on the third control and takes values between 27 and 29 Hz.

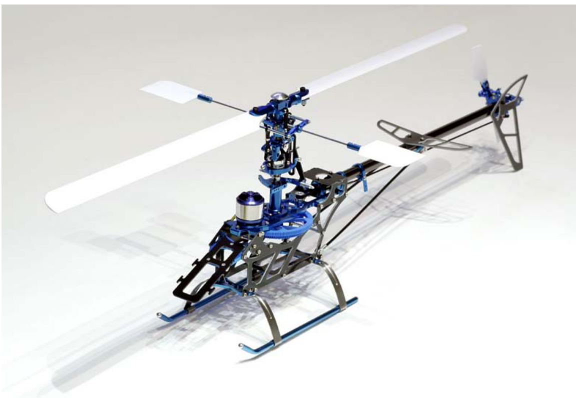

Figure 1: Electrical helicopter Walkera 450.

There are two types of observation: the TV tracking system and the onboard sensor.

1.1 TV observation

The TV observation is based on two standard Web cameras connected to computer through the USB ports. The cameras can be installed at arbitrary places at the distance of 2-4 meters from the helicopter. No calibration or position adjustment is needed. 4 small diodes were attached to the helicopter, and the distances between them were measured and known by the computer. This information is enough for calculation of the full helicopter state vector by one image in a camera coordinate system. The spots from 4 diodes on the screen are recognized, their centers and shapes are

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calculated. It is not a complicated geometrical task to find three dimensional coordinates of a tetrahedron tops by their two dimensional projections on the screen.

The navigation system consists of two parts: primary adjustment and tracking. Primary adjustment means calculation of the mutual positions of cameras including focus coordinates and Euler angles. The input is only one image of the diodes tetrahedron in each camera.

The helicopter state vector was converted to the earth coordinate system. The origin of this system is a point in the middle between the two camera focuses. The focuses lies in the OYZ plane while the axis OY is vertical. The vertical direction is recognized in the image by an additional vertical line with two diodes installed in the room. The axis OX is directed to the half space of the helicopter.

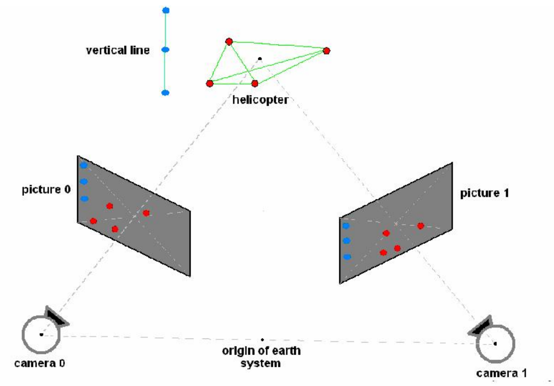

Figure 2: TV observation

Tracking is made by the Extended Kalman-Bucy filter. Camera position parameters are estimated with errors that must be corrected during observation. These parameters are included in the full state vector of the system which contains 20 variables. Coefficients of the linearized plant equations are calculated near the balanced realization with zero linear and angular velocities.

2. Sensor noises

An onboard sensor contains three gyroscopes and three acceleration gauges.

The gyroscopes measure angular velocities of the helicopter. The range between -3 and +3 rad/s is covered by a grid with 1024 points. The sample rate is around 273.5 samples per second. It corresponds to 10 samples per revolution of the main rotor.

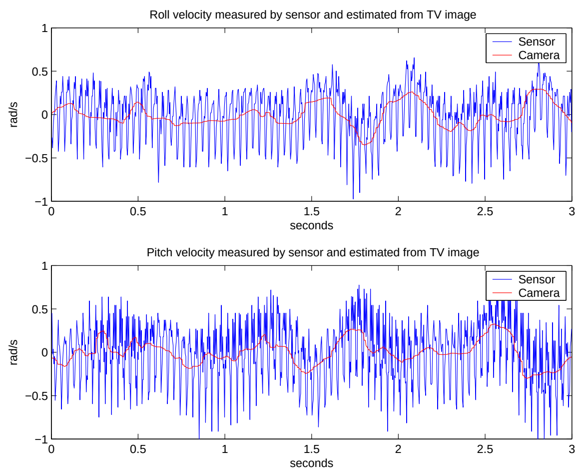

Figure 3: Angular velocities measured by the onboard sensor and by TV cameras

The data from gyroscopes show big and fast oscillations of fuselage forced by the rotation of the main rotor. Angles variations are small but the angular velocities of oscillations appear to be 3–4 times greater than their smoothed values. The Pitch and Roll velocities of the helicopter are shown in Fig. 3. The sensor data are blue, Kalman filter estimates by camera observations are red.

Typical spectra of the onboard sensor data from the Roll and Pitch gyroscopes are shown in Fig. 4. The spectra contain small values for the frequency less than 2 Hz, that express angular motion of the helicopter. The highest peak corresponds to the frequency of the main rotor rotation. The second, third and fourth harmonics are clearly seen. The last harmonic near the Nyquist frequency of 137 Hz corresponds to rotation of the tail rotor.

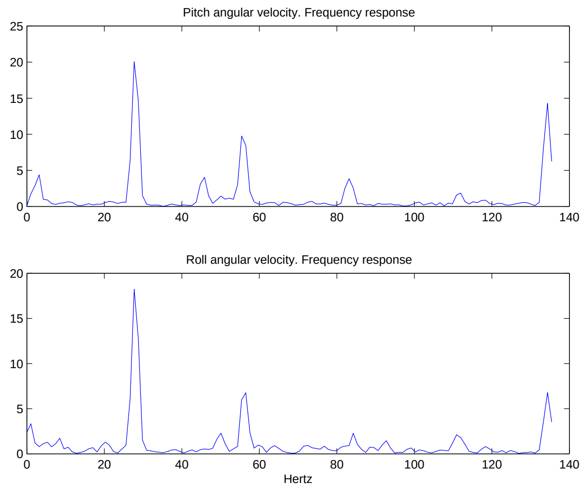

Figure 4: Frequency response of the body oscillation

A special "bells" technique[5,6] is supplied for precise estimation of frequencies and phases of the locally harmonic signal. With this technique the main rotor revolution time is estimated up to 1%. Estimates of frequency of the main rotor rotation on the time interval of 37 s is shown on Fig. 5. The gyroscope measurements in the roll and pitch channels were processed independently. It follows from Fig. 5 that the results coincide and accuracy is high. Even fluctuations of the main rotor frequency were the same. The reasons of the main rotor frequency fluctuations include control signals and an onboard battery charge trend.

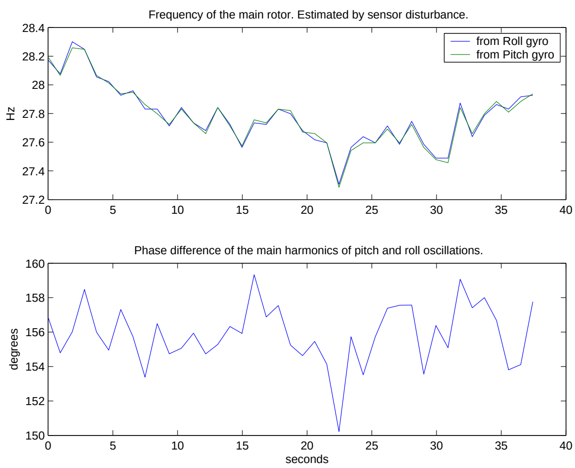

Figure 5: Estimates of frequency and phase of the main rotor by two gyroscopes

A detailed study of the oscillation signal have shown that it is a sum of pure harmonics with frequencies multiple to the main rotor rotation frequency F. Variations of the frequency F are shown in Fig. 5. All the harmonics were extracted from the signal by corresponding filtering. The biggest amplitude of the harmonics is achieved on the frequency F. The phases of the main harmonic were estimated on the sequential intervals of 256 samples. The roll and pitch signals were processed separately. The phase difference of the main harmonic estimated by roll measurement and by pitch measurement is shown in the second subplot of Fig. 5. It is seen that the phase difference is nearly the same for all frequencies of the main rotor. A deviation from the central value of 155° is less than 5°. Phases are very sensitive to the frequency errors. Therefore, small deviations of phases prove a high accuracy of the main rotor frequency estimate. The constant delay of 155° can be derived only from mechanics and aerodynamics of the main rotor.

3. Main rotor

A main rotor with the Hiller hub includes two blades and two servo blades. Position of the swash plate determines the pitch angle of the servo blade rod while the flapping angle of the servo blade rod determines the pitch angle of blades. The contribution of our investigation consists of a precise estimation of the approximation error for all forces and torques. A lightweight helicopter has specific values of normalized dimensionless parameters, many of them are close to 0.1. Any polynomial expression of these parameters is a sum terms with corresponding decimal orders. All forces and torques of the main rotor are approximated up to the second order of the approximation error.

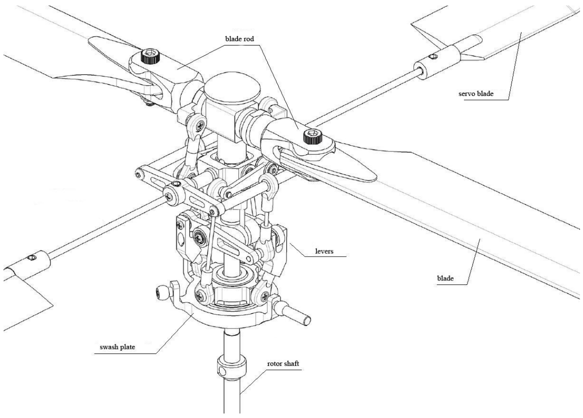

Fig. 1. The main rotor hub of the Hiller type

The Hiller hub is not equipped with vertical hinges. Therefore, the motion of blades and servo blades is flapping only in the planes that contain the main rotor shaft. The flapping angle of the servo blade rod is connected to the pitch angle of the blade rod. Therefore, the flapping motion of the servo blades determines the flapping motion of blades. But the inverse influence is not commonly studied because it is negligible for big machines. The results of this paper shows that this is not the case for small helicopters.

The basic equations of the main rotor theory can be found in.[1,2] This theory was successfully implemented in the helicopter autopilot system by TV observations.[3,4]

4. Full model of the flapping motion

Full dynamical models and their approximations are derived in this section for flapping motion of the servo blade rod and of the blade rod. The models are similar, therefore we consider a general rod motion instead of particular a blade or a servo blade. The motion is determined by the outer torques and the Coriolis forces and torques.

Consider a rod which is connected to the rotor shaft and rotates at the constant angular velocity ω_0. The helicopter rotates in the earth system of coordinates with the angular velocity $\Omega(t)$. The rod is attached to the shaft in a fixed point and it can move in the plane containing the shaft. The rod can also rotate with respect to its axis.

Assume an inertial system of coordinates is fixed, and the vector components in this system will be denoted by $^{\text{inert}}$, that is, $r^{\text{inert}} = \text{col}(r_x^{\text{inert}}, r_y^{\text{inert}}, r_z^{\text{inert}})$. The following four rotating systems of coordinates have the origins at the intersection of the rod and the shaft.

A helicopter system of coordinates r^h rotates with the angular velocity $\Omega(t)$ with respect to the inertial system. A shaft system of coordinates r^s rotates around the rotor shaft at the constant angular velocity ω_0. The rotation angle is denoted by $\xi(t) = \xi_0 + \omega_0 t$. A rod pivot system of coordinates r^b makes flapping motions with respect to the shaft system of coordinates. The flapping angle in the plane OYZ is denoted by $\beta(t)$. A rod system of coordinates r^0 rotates around the rod axis by the angle of attack $\phi(t)$.

Rotation transformations are defined by the following matrices:

$$r^h = Vr^s, \qquad r^s = Wr_b, \qquad r^b = Xr^0,$$

where the matrices take the form

$$X = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \qquad V = \begin{pmatrix} \cos\xi & 0 & -\sin\xi \\ 0 & 1 & 0 \\ \sin\xi & 0 & \cos\xi \end{pmatrix}.$$

The rod is influenced by the following forces and torques:

- The torque M of outer forces.
- The value of the stiffness torque $m_z^\omega(t)$ in the plane OYZ of the shaft coordinate system. This torque provides a uniform rotation of the shaft.
- The value of the stiffness torque $m_z^b(t)$ in the plane OXY of the rod system of coordinates. It provides the predetermined pitch angle of the rod. The torque $m_z^b(t)$ is applied for servo blades only.

The vectors of the stiffness torques are defined as follows. The torque $M_\omega^{\mathrm{inert}}(t)$ stabilizes the rotation in the inertial system of coordinates, while the torque $M_\phi(t)$ of implementation of the swash plate commands is defined in the rod system of coordinates:

$$M_{\omega,\mathrm{inert}}^{\mathrm{inert}}(t) = \begin{pmatrix} 0 \\ m_y^\omega(t) \\ 0 \end{pmatrix}, \qquad M_\phi^0(t) = \begin{pmatrix} 0 \\ 0 \\ m_z^b(t) \end{pmatrix}.$$

4.1 The general equation of the rod motion

The angular velocity vector δ is defined in the inertial system of coordinates. Its representation in the rod system of coordinates is

$$\delta^0 = -\dot\phi e_3 - \dot\beta X^T e_1 - \omega_0 X^T W^T e_2 + X^T W^T V^T \Omega,$$

where and further (e_1, e_2, e_3) denotes the standard basis vectors in R^3.

The standard mechanical equation of the rotation of a rigid body in the space gives

$$I^0 \dot\delta^0 = -\delta^0 \times (I^0 \delta^0) + M_\omega^0 + M_\phi^0 + M^0,$$

where

$$M_\omega^0 = m_z^b e_3, \qquad M_\phi^0 = m_y^\omega X^T W^T e_2,$$

and the torque M_ϕ is not applied to blades. Denote the rod matrix of inertia by

$$I^0 = \begin{pmatrix} i_x & 0 & -i_{xz} \\ 0 & i_x + i_z & 0 \\ -i_{xz} & 0 & i_z \end{pmatrix}.$$

A particular feature of the rod is a symmetric mass distribution with respect to the OY axis. The matrix I can be described by the main value of i_x and the coefficients

$$\kappa_{xz} = \frac{i_{xz}}{i_x}, \qquad \kappa_z = \frac{i_z}{i_x}.$$

Normal values for these coefficients are $\kappa_{xz,g} \approx -0.009$, $\kappa_{z,g} \approx 0.0016$ for servo blades, and $\kappa_{xz,g} \approx 0.0044$, $\kappa_{z,g} \approx 0.00085$ for blades.

Assume the helicopter angular velocity Ω is given Then the system of three equations for the servo blades contains three unknowns: the flapping angle β and the stiffness torques m_z^b and m_y^ω. After integration of this differential equation, the stiffness torque m_z^b is added to the outer torques applied to the blades.

The blade equation system contains also three unknowns. They are: the flapping angle β, the pitch angle ϕ and the stiffness torque m_y^ω. It is required to calculate the flapping motion angle of $\beta(t)$ for blades and the flapping motion angle of $\beta_g(t)$ for sevo blades.

4.2 Equation for the flapping motion

The stiffness torques of the rigid body m_z^b and m_y^ω are eliminated from the three equations of flapping motion. This can be done from the kinematic relation

$$e_1^T X[m_z^b e_3 + m_y^\omega X^T W^T e_2] = 0.$$

Long algebraic transformations lead to the following formula:

$$\begin{aligned}
&\frac{1}{i_x} e_1^T X[I^0 \dot{\delta}^0 + \delta^0 \times (I^0 \delta^0)] = -\ddot{\beta} - \omega_0^2 \cos\beta \sin\beta + 2\omega_0 \Omega_z^b \cos\beta + \dot{\Omega}_x^b - \Omega_y^b \Omega_z^b \\
&+\kappa_{xz} \Bigg\{ \cos\phi \Big[\ddot{\phi} + 2\omega_0 \Omega_x^b \cos\beta - \dot{\Omega}_z^b - \Omega_x^b \Omega_y^b \Big] + \sin\phi \Big[(\omega_0 \cos\beta - \Omega_y^b)^2 - (\dot{\phi} + \omega_0 \sin\beta - \Omega_z^b)^2 \Big] \Bigg\} \\
&+\kappa_z \sin\phi \Bigg\{ \cos\phi \Big[-2\dot{\phi}\dot{\beta} + 2\dot{\phi}\Omega_x^b + 2\omega_0 \Omega_x^b \sin\beta + \dot{\Omega}_y^b - \Omega_x^b \Omega_z^b \Big] \\
&+ \sin\phi \Big[-\ddot{\beta} + \omega_0^2 \sin\beta \cos\beta + 2\omega_0 \dot{\phi} \cos\beta - 2\dot{\phi}\Omega_y^b - 2\omega_0 \Omega_y^b \sin\beta + \dot{\Omega}_x^b + \Omega_y^b \Omega_z^b \Big] \Bigg\}.
\end{aligned}$$

This expression must be equal to the outer torque divided by the inertia matrix:

$$\frac{1}{i_x} e_1^T X[I^0 \dot{\delta}^0 + \delta^0 \times (I^0 \delta^0)] = \frac{1}{i_x} M_x^b = \frac{1}{i_x} M_x^s.$$

The torque m_z^b is considered as an outer torque for the blades. The pitch angle ϕ remains unknown. To find it, an additional equation is to be derived from the general rotation equation by left multiplication by $e_3^T W X$:

$$e_3^T W X[I^0 \dot{\delta}^0 + \delta^0 \times (I^0 \delta^0)] = e_3^T W X M^0 = M_z^s.$$

Algebraic transformations that lead to the equation

$$\frac{1}{i_x} e_3^T W X[I^0 \dot{\delta}^0 + \delta^0 \times (I^0 \delta^0)] = E_0 + \kappa_{xz} E_{xz} + \kappa_z E_z,$$

where

$$\begin{aligned}
E_0 &= \sin\beta \Big[-\dot{\Omega}_y^b - 2\omega_0 \dot{\beta} \sin\beta + \Omega_z^0 (2\dot{\beta} - \Omega_x^b) \Big], \\
E_{xz} &= \ddot{\phi} \sin\phi \sin\beta + \ddot{\beta} \cos\beta \cos\phi - \dot{\Omega}_x^0 \cos\beta - \dot{\Omega}_z^0 \sin\phi \sin\beta + 2\omega_0 \dot{\phi} \cos\phi \sin^2\beta \\
&\quad +4\omega_0 \dot{\beta} \sin\phi \cos\beta \sin\beta + (\dot{\phi}^2 - \dot{\beta}^2 + \omega_0^2) \cos\phi \sin\beta + \Omega_z^0 [-2\dot{\phi} \cos\phi \sin\beta - 2\dot{\beta} \sin\phi \cos\beta - 2\omega_0 \cos\phi \\
&\quad +\Omega_y^0 \cos\beta + \Omega_z^0 \cos\phi \sin\beta] + \Omega_x^0 [2\dot{\beta} - \Omega_x^b] \sin\beta, \\
E_z &= -\ddot{\phi} \cos\beta + \ddot{\beta} \cos\phi \sin\phi \sin\beta - \dot{\Omega}_y^0 \cos\phi \sin\beta + \dot{\Omega}_z^0 \cos\beta + 2\dot{\phi}\dot{\beta} \cos^2\phi \sin\beta \\
&\quad -2\omega_0 \dot{\phi} \cos\phi \sin\phi \cos\beta \sin\beta - 2\omega_0 \dot{\beta} \sin^2\phi \cos^2\beta - (\omega_0^2 - \dot{\beta}^2) \cos\phi \sin\phi \cos\beta \\
&\quad +\Omega_x^0 [-2\dot{\phi} \cos\phi \sin\beta - 2\dot{\beta} \sin\phi \cos\beta - 2\omega_0 \cos\phi + \Omega_y^0 \cos\beta + \Omega_z^0 \cos\phi \sin\beta].
\end{aligned}$$

5. Cross influence of blade and servo blade subsystems

The angles of the swash plate, of the servo blade and blade pitches and of the servo blade flapping motions are connected by kinematic conditions. In the sequence, all variable related to the servo blades are provided with the superscript g to distinguish them from the blades.

A direction will be defined with respect to some fix axis which is commonly directed to the air flow. The angle between a direction of a rod and this axis is called an azimuth. It is calculated in the plane OXZ of the speed system of coordinate and in the negative direction, from OX to OZ.

The cyclic pitch of the swash plate is determined by the coefficients Θ_1 and Θ_2 that together form the complex number $\Theta_1 + i\Theta_2$. The angle Θ_1 corresponds to the zeros azimuth and the angle Θ_2 corresponds to $-\pi/2$. The swash plate position at the azimuth ψ is

$$\theta(\psi) = \mathrm{Im}(e^{-i\psi}\Theta),$$

The pitch angle of the servo blade at the azimuth ψ is defined by

$$\phi_g(\psi) = D_u\theta(\psi + \pi/2) = -D_u \operatorname{Re}(e^{-i\psi}\Theta),$$

where $D_u = 1.6$ is the transfer coefficients.

Set one of the blades to be the leading blade, and its azimuth will be denoted by ψ. The flapping angle and the pitch angle of this blade are denoted by$\beta(\psi)$ and $\phi(\psi)$, respectively. This leading blade indicates a servo blade that goes ahead by $90°$. Denote its flapping angle by $\beta_g(\psi{+}\pi/2)$.

The kinematic relation between the angles is determined by the hub construction and can be described by the equation

$$\begin{aligned}\phi(\psi) &= \phi_0 + D_g\beta_g(\psi{+}\pi/2) + D_\Theta\theta(\psi + \pi/2)\\ &= \phi_0 + D_g\beta_g(\psi{+}\pi/2) - D_\Theta \operatorname{Re}(e^{-i\psi}\Theta),\end{aligned}$$

where ϕ_0 is the collective pitch, $D_g = 0.8$ and $D_\Theta = 0.5$ are the fixed transfer coefficients.

The kinematic condition on the blade angle of pitch ϕ and the servo blade flapping angle β_g forces the torques on a servo blade and on a blade

$$M^b_{\text{hub}}(\psi) = \begin{pmatrix}0\\0\\m_{b,g}(\psi)\end{pmatrix},\ M^s_{\text{hub},g}(\psi+\pi/2) = \begin{pmatrix}-m_{b,g}(\psi)\\0\\0\end{pmatrix}.$$

The positive value of $m_{b,g}(\psi)$ means that the torque acts to the decreasing of the blade pitch $\phi(\psi)$ and the torque on the servo blade tries to increase the flapping angle $\beta_g(\psi + \pi/2)$.

6. Principle parts in equations

Simplification of the general motion equations can be achieved by expansion of the nonlinear functions in the power series with respect of all variables and by taking the main polynomial parts. It was recognized that all dimensionless variables can be divided in a number of groups by their decimal values. The group of the order n contains all variables that have maximal values not greater than $\alpha 10^{-n}$ with the coefficient $\alpha \le 2$, and not belong to the group $n + 1$. If a variable f has an order n then the variable $|f|^{1/n}$ has an order 1.

The next list contains variable which were decided to have the order 1. All angles are calculated in radians, all angular velocities are divided by ω_0.

$$\begin{aligned}\varepsilon &= \operatorname{col}(\beta, \beta_g, \phi, \phi_g, \bar\Omega_x, \bar\Omega_z, |\bar\Omega_y|^{1/2}, |\bar\Omega'_{xz}|^{1/3}, |\bar\Omega'_y|^{1/4},\\ &\quad \kappa_{xz}^{1/2}, \kappa_z^{1/3}, \kappa_{xz,g}^{1/2}, \kappa_{z,g}^{1/3}, \kappa_i),\end{aligned}$$

where $\kappa_i = i_{x,g}/i_x \approx 0.12$.

For any function $f(t)$ define the normalized derivative as $f'(t) = \omega_0^{-1}\dot f(t)$.

Extract the terms in the general flapping motion equations which have the minimal order n and the next order $n + 1$. The rest is residual denoted by $\mathcal{O}(|\varepsilon|^{n+2})$. After some algebra the general equations become much simpler:

$$-\beta'' - \beta + 2\bar\Omega^s_z = \frac{1}{i_x\omega_0^2}M^b_x + \mathcal{O}(|\varepsilon|^3), \qquad -\beta''_g - \beta_g + 2\bar\Omega^s_z = \frac{1}{i_{x,g}\omega_0^2}M^b_{x,g} + \mathcal{O}(|\varepsilon|^3).$$

The equation for the angle of blade pitch is

$$\beta[-2\beta'\beta + 2\beta'\bar\Omega^b_z - \bar\Omega^b_x\bar\Omega^b_z] + \kappa_z[-\phi'' - \phi - 2\bar\Omega^b_x] = \frac{1}{i_x\omega_0^2}(M^s_z + \kappa_{xz}M^b_x) + \mathcal{O}(|\varepsilon|^5).$$

The stiffness torque m^ω must be eliminated from the equations. This lead to the system of two differential equations and one algebraic equation with respect to the flapping angles β, β_g and the pitch of attack ϕ:

$$\begin{aligned}-\beta'' &= \beta - 2\bar\Omega^s_z + \frac{1}{i_x\omega_0^2}(M^s_{x,\text{air}} + M^s_{x,e}) + \mathcal{O}(|\varepsilon|^3),\\ \phi(\psi) &= \phi_0 + D_g\beta_g(\psi{+}\pi/2) - D_\Theta \operatorname{Re}(e^{-i\psi}\Theta),\\ -\beta''_g(\psi+\pi/2) - \beta_g(\psi+\pi/2) &= 2\bar\Omega^s_x(\psi) + \kappa_i^{-1}\beta(\psi)[2\beta'(\psi)\beta(\psi) - 2\beta'(\psi)\bar\Omega^s_z(\psi) + \bar\Omega^s_x(\psi)\bar\Omega^s_z(\psi)]\\ &\quad + \frac{\kappa_{xz}}{\kappa_i}(-\beta''(\psi) - \beta(\psi) + 2\bar\Omega^s_z(\psi)) + \frac{M^s_{x,\text{air},g}(\psi+\pi/2)}{i_{x,g}\omega_0^2} + \mathcal{O}(|\varepsilon|^5).\end{aligned}$$

7. Principle parts of the main rotor forces and torques

Similar approximations can be done in the standard theory of the main rotor. The equations can be truncated with remaining the principle part and the next order part.

Introduce the following notation.

ρ - density of air; b - blade chord, c_y^α - section lift coefficient, c_x - blade drag coefficient, ω - rotational frequency of main rotor, R - blade length, B - tip loss factor, λ - rotor induced inflow ratio, Θ_1 - longitudinal cyclic pitch, Θ_2 - lateral cyclic pitch, ϕ_0 - blade collective pitch, S - rotor side force, $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ - angular velocity of helicopter, $V = (V_x, V_y, V_z)$ - linear velocity of main rotor hub, (a_1, b_1) - coefficients of flapping motion of blade, (a_{1g}, b_{1g}) - coefficients of flapping motion of servoblade, k_e - coefficient of elastic moment of blade, $r_{1,g}, r_{2,g}$ - minimal and maximal radii of servoblade, D_Θ - transfer coefficient from swash plate angle to blade section angle of attack, D_u - transfer coefficient from swash plate angle to servoblade section angle of attack, D_g - transfer coefficient from servoblade flapping angle to blade section angle of attack, γ_* - blade mass characteristic, γ_{*g} - servoblade mass characteristic.

The helicopter dynamics equations have a specific form that is isomorphic to arithmetic of the complex numbers. Therefore, introduce the corresponding complex values: $\mathsf{a} = a_1 + ib_1$ - complex coefficient of flapping motion of blade, $\mathsf{a}_g = a_{1,g} + ib_{1,g}$ - complex coefficient of flapping motion of servoblade, $\Theta = \Theta_1 + i\Theta_2$ - complex coefficient of the swash plate position, $\mathsf{D}_\Theta = D_g\mathsf{a}_g + iD_\Theta\Theta = D_{a\Theta} + iD_{b\Theta}$ - complex coefficient from the blade position to the section angle of attack, $\mathsf{V}_{xz} = V_x + iV_z$. The coordinate system is shown in Fig. 1.

Dimensionless values are denoted by bar. Linear velocity: $\bar{V} = V/(\omega R)$. Angular velocity: $\bar{\Omega} = \Omega/\omega$. Length: $\bar{r} = r/R$. Additional notation for the following small values:

$$B_{2,g} = \frac{r_{2,g}^2 - r_{1,g}^2}{2R^2}, \qquad B_{2,g} = \frac{r_{2,g}^4 - r_{1,g}^4}{4R^4},$$

Define the coefficients

$$\begin{aligned} \gamma_* &= \frac{\rho b c_y^\alpha R^4}{2I_{\rm h}}, \quad k_\gamma = \frac{4k_e}{\rho b c_y^\alpha \omega^2 R^4 B^4}, \\ \mathsf{D}_\Theta &= D_g\mathsf{a}_g + iD_\Theta\Theta = D_{a\Theta} + iD_{b\Theta}. \end{aligned}$$

The list ε of the first order variables from the previous section is supplemented by

$$\Delta\varepsilon = (\lambda, \Theta, c_x^{1/2}, c_{xg}^{1/2}, B_{2,g}^{-1/2}, B_{4,g}^{-1/2}, \phi_0, |\bar{\mathsf{V}}_{xz}|).$$

1. Aerodynamic force of the main rotor blades normal to the disk plane:

$$\begin{aligned} T &= \rho b c_y^\alpha \omega^2 R^3 \Bigg\{ \frac{B^2}{2}\lambda(1-\bar{\Omega}_y) + \phi_0\left(\frac{B^3}{3}(1-\bar{\Omega}_y)^2 + \frac{B}{2}|\bar{\mathsf{V}}_{xz}|^2\right) \\ &\quad + \frac{B^2}{4}\,\mathrm{Re}\left[(\bar{\mathsf{V}}_{xz})^*\Big(2\mathsf{D}_\Theta(1-\bar{\Omega}_y) - \bar{\Omega}_{xz} - \mathsf{a}\bar{\Omega}_y\Big)\right] + \mathcal{O}(|\varepsilon|^3)\Bigg\}. \end{aligned}$$

2. Pitching and rolling moments of the main rotor blades:

$$\mathsf{M}_T = \frac{1}{2}k_e i\mathsf{a}.$$

3. Torque of the main rotor blades:

$$M_k = \rho b\omega^2 R^4 c_y^\alpha \Big\{ m_{k,2} + m_{k,3} + \mathcal{O}(|\varepsilon|^4) \Big\},$$

where $m_{k,2}$ and $m_{k,3}$ are of the second and the third orders

$$\begin{aligned} m_{k,2} &= \frac{1}{4}\frac{c_x}{c_y^\alpha} - \frac{B^3}{3}\phi_0\lambda + \frac{B^4}{8}\,\mathrm{Re}[\mathsf{D}_\Theta(\mathsf{a}+\bar{\Omega}_{xz})^*] - \frac{B^2}{2}\lambda^2 - \frac{B^4}{8}|\mathsf{a}+\bar{\Omega}_{xz}|^2, \\ m_{k,3} &= -\bar{\Omega}_y\left[\frac{1}{2}\frac{c_x}{c_y^\alpha} - \frac{B^3}{3}\phi_0\lambda + \frac{B^4}{8}\,\mathrm{Re}(\mathsf{D}_\Theta(\mathsf{a}+\bar{\Omega}_{xz})^*)\right] + \mathrm{Re}(\bar{\mathsf{V}}_{xz})^*\left[\frac{B^2}{4}\phi_0\bar{\Omega} - \frac{B^2}{4}\lambda\mathsf{D}_\Theta - \frac{B^2}{2}\lambda\mathsf{a}\right]. \end{aligned}$$

4. Coefficients of the flapping motion of blades:

$$\mathsf{a} \;=\; \frac{1}{\Delta}\left[\mathsf{D}_\Theta - \bar\Omega_{xz} + \frac{8}{B^4\gamma_*} i\bar\Omega_{xz} + \left(\frac{8}{3B}\phi_0 + \frac{2}{B^2}\lambda\right)\bar{\mathsf{V}}_{xz} + (\bar\Omega_{xz} - 2\mathsf{D}_\Theta)\bar\Omega_y\right] + \mathcal{O}(|\varepsilon|^3),$$

where

$$\Delta = (1 + k_\gamma i) - \left(1 + \frac{8}{B^4\gamma_*} i\right)\bar\Omega_y.$$

5. Coefficients of the flapping motion of servoblades:

$$\begin{aligned}\mathsf{a}_g \;=\;& -\bar\Omega_{xz} + iD_u\Theta + \frac{2}{\gamma_{*g}B_{4,g}} i\bar\Omega_{xz} + \frac{-\bar\Omega_y\left[4\gamma_{*g}^{-2}B_{4,g}^{-1}\bar\Omega_{xz} + (2\gamma_{*g}^{-1} + B_{4,g}i)D_u\Theta\right]}{B_{4,g} - \bar\Omega_y(B_{4,g} + 2\gamma_{*g}^{-1}i)}\\ &+ \frac{B_{2,g}(\lambda\bar{\mathsf{V}}_{xz} + \mathcal{O}(|\varepsilon|^3))}{B_{4,g} - \bar\Omega_y(B_{4,g} + 2\gamma_{*g}^{-1}i)}.\end{aligned}$$

6. Aerodynamic force of the servo blades normal to the disk plane:

$$T_g \;=\; \rho b_g c_{yg}^\alpha \omega^2 R^3 B_{2,g}\left\{\lambda(1 - \bar\Omega_y) - D_u\,\mathrm{Im}((\bar{\mathsf{V}}_{xz})^*\Theta) - \frac{1}{2}\mathrm{Re}((\bar{\mathsf{V}}_{xz})^*\bar\Omega_{xz}) + \mathcal{O}(|\varepsilon|^3)\right\}.$$

7. Torque of the servoblades:

$$M_{k,g} = \rho b_g c_{yg}^\alpha \omega^2 R^4\left\{m_{k,2,g} + m_{k,3,g} + \mathcal{O}(|\varepsilon|^4)\right\},$$

where

$$\begin{aligned}m_{k,2,g} \;&=\; B_{4,g}\frac{c_{x,g}}{c_{yg}^\alpha} - B_{2,g}\lambda^2 - \frac{1}{2}B_{4,g}|\mathsf{a}_g + \bar\Omega_{xz}|^2 + \frac{1}{2}B_{4,g}D_u\,\mathrm{Im}((\Theta)^*(\mathsf{a}_g + \bar\Omega_{xz})),\\ m_{k,3,g} \;&=\; \bar\Omega_y\left(-2B_{4,g}\frac{c_{x,g}}{c_{yg}^\alpha} + \frac{1}{2}B_{4,g}D_u\,\mathrm{Im}(\Theta(\mathsf{a}_g + \bar\Omega_{xz})^*)\right) - B_{2,g}\lambda\,\mathrm{Re}\left[(\bar\Omega_{xz})^*\left(\mathsf{a}_g + \frac{1}{2}iD_u\Theta\right)\right].\end{aligned}$$

8. Longitudinal and lateral forces of the main rotor blades:

$$\mathsf{B} = \mathsf{B}_T + \mathsf{B}_Q = \rho b c_y^\alpha \omega^2 R^3\left\{\mathsf{b}_2 + \mathsf{b}_3 + \mathcal{O}(|\varepsilon|^4)\right\},$$

where b_2 and b_3 are the second order and the third order terms,

$$\begin{aligned}b_2 \;&=\; \left(\frac{B^3}{3}\phi_0 + \frac{3B^2}{4}\lambda\right)\mathsf{a} - \frac{B^2}{4}\lambda\mathsf{D}_\Theta + \left(\frac{B^3}{6}\phi_0 + \frac{B^2}{2}\lambda\right)\bar\Omega_{xz},\\ b_3 \;&=\; \frac{1}{2}\bar{\mathsf{V}}_{xz}\frac{c_x}{c_y^\alpha} - \frac{B}{2}\bar{\mathsf{V}}_{xz}\lambda\phi_0 - \bar\Omega_y\left(\frac{B^2}{4}\lambda\mathsf{a} + \frac{B^3}{3}\phi_0\mathsf{a} + \frac{B^3}{6}\phi_0(\mathsf{a} - \bar\Omega_{xz}) + \frac{B^2}{4}\lambda\mathsf{D}_\Theta\right)\\ &+\frac{B^2}{8}\bar{\mathsf{V}}_{xz}\left(|\mathsf{a}|^2 + i\,\mathrm{Im}(\mathsf{a}(2\mathsf{D}_\Theta - 3\bar\Omega_{xz})^*) + (\mathsf{D}_\Theta)^*\mathsf{a} + \mathrm{Re}((\mathsf{D}_\Theta)^*\bar\Omega_{xz})\right) + \frac{B^2}{8}(\bar{\mathsf{V}}_{xz})^*(\mathsf{a}+\mathsf{D}_\Theta)(\mathsf{a}+\frac{1}{2}\bar\Omega_{xz}).\end{aligned}$$

9. Longitudinal and lateral forces of the servoblades:

$$\mathsf{B}_g = \rho b_g c_{yg}^\alpha \omega^2 R^3 B_{2,g}\left\{\mathsf{b}_{g,2} + \mathsf{b}_{g,3} + \mathcal{O}(|\varepsilon|^4)\right\},$$

where

$$\begin{aligned}\mathsf{b}_{g,2} \;&=\; \lambda\left(\frac{3}{2}\mathsf{a}_g + \bar\Omega_{xz} - \frac{1}{2}iD_u\Theta\right),\\ \mathsf{b}_{g,3} \;&=\; \frac{c_{x,g}}{c_{yg}^\alpha}\bar{\mathsf{V}}_{xz} + \frac{1}{2}\lambda\Omega_y(-\mathsf{a}_g + iD_u\Theta) + \frac{1}{4}\bar{\mathsf{V}}_{xz}\left[|\mathsf{a}_g|^2 - 2i\,\mathrm{Re}\left(\left(D_u\Theta + \frac{1}{2}i\bar\Omega_{xz}\right)(\mathsf{a}_g)^*\right)\right.\\ &\left.+2i\,\mathrm{Im}(\bar\Omega_{xz}(\bar{\mathsf{a}}_g)^*) + D_u\,\mathrm{Im}((\Theta)^*\bar\Omega_{xz}) - iD_u(\Theta)^*\mathsf{a}_g\right]\\ &+\frac{1}{4}(\bar{\mathsf{V}}_{xz})^*\left[\frac{1}{2}iD_u(\Theta\bar\Omega_{xz}) + i\left(D_u\Theta + \frac{1}{2}i\bar\Omega_{xz}\right)\mathsf{a}_g + \mathsf{a}_g(\mathsf{a}_g + \bar\Omega_{xz})\right].\end{aligned}$$

8. Conclusion

The full set of nonlinear equations were presented for the flapping motion of blades and servo blades in the Hiller hub. The equation were simplified by extraction of the main part with respect to dimensionless parameters taking values around 0.1. It was shown that the stiffness torque between flapping motion of the servo blades and the blade pitch of attack is not negligible. A regulator in the autopilot system was designed on the basis of the derived equations.

References

[1] Mil M.L., Nekrasov A.V., Braverman A.S., Grodko L.N., Leikand M.A.. (1966). *Helicopters*. Designing and calculations. Moscow.

[2] Johnson W. (1980). *Helicopter theory*. Princeton Univ. Press.

[3] Barabanov A.E., Vazhinsky N.Yu., Romaev D.V. (2007). Full autopilot for small electrical helicopter. *Proc. of the 33th European rotorcraft forum*. Kazan. September 11-13.

[4] Barabanov A.E., Romaev D.V. Helicopter Modeling and Autopilot Design. *Proc. of the 18th IFAC World Congress*, Milan, Italy, 2011.

[5] A.E.Barabanov. Fast identification of the voiced speech signal. *Swedish-Russian Control Conference*, St. Petersburg, 2011.

[6] A.E.Barabanov. Identifiction of parameters of a polyharmonic model of a speech signal. *Proc. of the XII Russian control conference*, p. 3038-3049, 2014.