# The Weakest-Link Models in the Strength and Reliability Analysis of the Rocket Fairing Shell Made of Glass Ceramics

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## Abstract
Based on the models of brittle failure, the strength-size scaling of the structural glass ceramic material has been studied, the size and distribution of flaws initiating material fracture have been evaluated, the safety factors of the components made from this material as a function of predetermined failure probability (required reliability), material structure homogeneity, component dimensions and material microstructure element size have been obtained.

## 1. Introduction

The use of the developed at ORPE "Technologiya" high-density glass-ceramic lithium-aluminum-silicate-based material OTM 357 as a structural material for highly loaded rocket fairings causes higher requirements placed upon its load-bearing capacity. The strength reliability of the specific fairing design depends on the material inhomogeneity extent which in its turn depends on the manufacturing methods responsible for various kinds of flaws. The investigation of glass ceramics strength from the viewpoint of statistical theory of brittle failure makes it possible to predict the failure probability and to substantiate the safety factor of the components made from this material. The statistical models of brittle failure strength are constructed under the assumption that the scatter in material strength values is a result of random inhomogeneity in the material and it depends on the distribution of flaw sizes in its microstructure. In this case the distribution of local strength values over the material volume elements is associated with the distribution of the most dangerous flaws over these elements.

## 2. Brittle failure models

Two models were used to describe statistically the strength of brittle materials: the Weibull model[1] based on asymptotic distribution of extreme values and the McClintock model[2] in which the relationship between the component size and the size of the material microstructure element is taken into account.
The Weibull model of strength distribution comes from an inverse power-series distribution of flaw sizes:

$$H(c) = 1 - k_c c^{-n}, \qquad (1)$$

where *H(c)* is the cumulative distribution function, *c* is the flaw size, $k_c$ and *n* are the distribution parameters. The exponent *n* the value of which describes the scatter in flaw sizes is related to the Weibull distribution modulus *m*.
The Weibull model enabling one to establish the relationship between the failure probability, stress and loaded with maximum stress material volume is most often used in the form of two-parameter distribution

$$P_f = 1 - \exp\left[ -\int_V \left( \frac{s}{s_c} \right)^m dV \right] = 1 - \exp\left[ -k_V V \left( \frac{s_{\max}}{s_c} \right)^m \right],$$

where *V* is the volume of the element under consideration, $\sigma_{\max}$ is the maximum tensile stress in this volume, $\sigma_c$ and *m* are the distribution parameters, $k_V$ is the coefficient describing the type of loading. The load factor $k_V$ is the measure of stress distribution uniformity. Under uniform tensile $k_V = 1$. Under pure flexure with the uniform distribution of stresses over the specimen length $k_V = [2(m + 1)]^{-1}$ and under three-point flexure $k_V = [2(m + 1)^2]^{-1}$.

From the two-parameter description of the statistical strength distribution the relationship between the strength and effective stressed volume (surface area) of the material can be determined, for instance, in the specimen and in the component

$$\bar{s}_1 / \bar{s}_2 = (V_{ef2} / V_{ef1})^{1/m}, \tag{2}$$

$$\bar{s}_1 / \bar{s}_2 = (S_{ef2} / S_{ef1})^{1/m}, \tag{3}$$

where $\bar{s}_1$, $V_{ef1}$ and $S_{ef1}$; $\bar{s}_2$, $V_{ef2}$ and $S_{ef1}$ are average strength, effective stressed volume and effective stressed surface area of the specimen and component, respectively. These relationships make it possible to recalculate strength values obtained with the use of different test schemes and specimens of different size depending on the location of the most dangerous flaws – in the volume or over the surface.
One of the approaches correlating the integrity of the component material with the typical size of flaws is the McClintock statistical model. It differs from the Weibull model in that it provides a possibility to substantiate the nature of "the weakest link" and to predict the influence of the ratio of ceramic component size to the size of ceramic material structure element on the strength. According to this model the probability of failure for the case when the stress is lower than σ is

$$P_f = 1 - \exp\left\{ -\frac{Aa}{S} \exp\left( -\frac{b}{(s / s_0)^2} \right) \right\}, \tag{4}$$

where $A$ is the surface area of the component, $S$ is the area of the material microstructure element, $a$ and $b$ are the distribution parameters, $s_0 \cong K_{IC} / \sqrt{2c}$ is the strength of the material with a crack having the length $c$, $K_{IC}$ is critical stress intensity factor.
The distribution parameters $a$ and $b$ are related to the probability ($w$) of the existence of flaw in the form of a crack of size $c$ by the following expressions:

$$a = \frac{\sqrt{w}}{0.5 - 1/\ln w}, \; b = -\ln w. \tag{5}$$

From equation (4) the expression for the strength with predetermined failure probability can be obtained

$$s(P_f) = s_0 \sqrt{\frac{b}{\ln\left( \frac{Aa}{S} \Big/ (-\ln(1 - P_f)) \right)}}. \tag{6}$$

Eq.6 is valid for sufficiently large surface areas and small values of probability $P_f$. For the use of the probabilistic approach to be valid the ceramic component surface area under consideration must contain about $10^3$ structure elements[3]. McClintock studied his model for $A/S$ from $1\times10^2$ to $1\times10^5$.

## 2. Investigation results

The distribution of strength values for three OTM 357 specimens of different size obtained in three-point flexure tests are shown in Weibull coordinate grid in Fig.1. Figure 2 shows a relative decrease in strength with a relative increase in volume and surface area described by strength-size effect equations (2) and (3) with Weibull moduli $m$ for effective volume and surface area equal to 12.8 and 8.5, respectively.
The distribution of flaws is represented by the Weibull distribution of strength in terms of the dispersion of strength or the value of the Weibull modulus $m$. Judging from the values $m$ obtained, the size and consequently the hazard chance of glass ceramics surface flaws have wider distribution than the size of volume defects.
The dependences of strength-size effect obtained make it possible to predict the average tensile strength of glass ceramics in the component with due regard for its effective volume (surface area) equivalent to the uniform tensile.

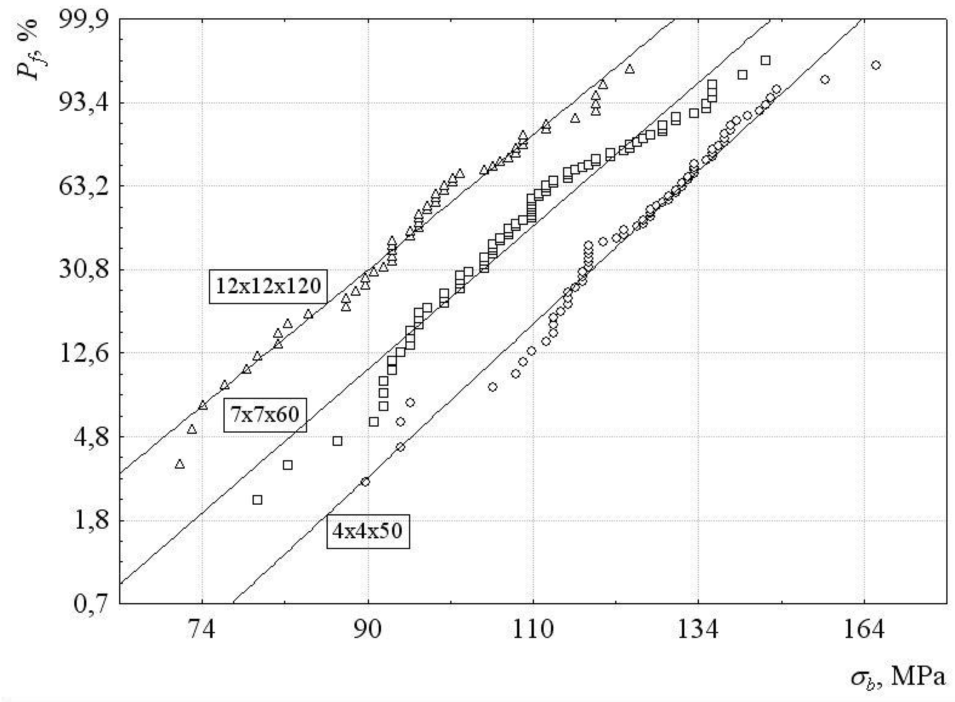


Figure 1: The distribution of flexural strength values ($\sigma_b$) for three different-size specimens made of glass ceramic material OTM 357

Thus, for a glass ceramic aerial fairing having $V_{ef} \approx 130{,}000$ mm$^3$ and $S_{ef} \approx 40{,}000$ mm$^2$ the values of the average tensile strength obtained from the values of flexural strength with the equation $\bar{s}_t = \bar{s}_b [2(m+1)^2]^{-1/m}$ and predicted with equations (1) and (2) are 48 and 34 MPa, respectively.

The results of fairing structure analysis by finite element method show that the maximum stress in the fairing shell material caused by applied aerodynamic loads does not exceed 18 MPa. Hence the predicted safety factor of the fairing structure material will be 2.6 in case of failure from the volume flaws and it will be 1.9 in case of failure from the surface flaws**.** This conclusion was supported in the full-scale destruction tests of fairings from the results of which the experimental values of the safety factor from 2.3 to 3.5 were obtained.

The probability of failure beginning from the surface depends both on the surface flaw size distribution parameter ***m*** and on the flaw depth $\Delta d$.

According to the model described by Evans[4] and based on the statistics of flaw size and location, the failure probability obtained at the stress intensity factor $K_I = K_{IC}$ is

$$P_f = 1 - \exp\left\{ -\left( \frac{s}{s_f} \right)^m \left[ \frac{1}{2^{m/2}} \left( \frac{V_{ef}}{V_0} \right) - \left( 1 - \frac{1}{2^{m/2}} \right) \frac{S_{ef} \Delta d}{V_0} \right] \right\},$$

where $s_f = K_I / \sqrt{pc}$, $c$ is the volume flaw size, $V_o$ is the elementary (of the same order with the flaw size) volume. The use of the last equation together with strength-size effect equation (2) makes it possible to evaluate the depth of a surface flaw $\Delta d$. In this case the relative rupture stress for two bodies of different effective size is

$$\left( \frac{\bar{s}_1}{\bar{s}_2} \right)^m = \frac{\dfrac{V_{ef2}}{2^{m/2}} - \left(1 - 2^{-m/2}\right) S_{ef2} \Delta d}{\dfrac{V_{ef1}}{2^{m/2}} - \left(1 - 2^{-m/2}\right) S_{ef1} \Delta d}.$$

Using the abovementioned data on the values of effective volumes and surfaces of the specimens and components and also calculated value of the Weibull modulus and solving this equation with reference to $\Delta d$, we obtain that a minimum size of flaws leading to the component failure is within 150 and 200 μm. This result goes with the statement[5] that according to the models of Griffith model type the size of flaws limiting the strength of most structural ceramic materials is from 5 to 200 μm.

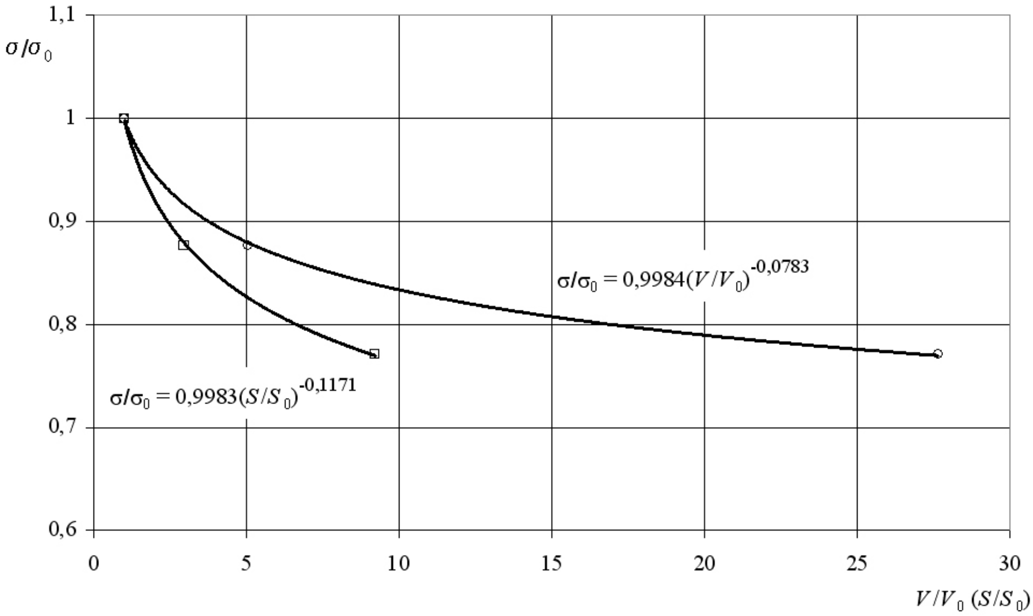


Figure 2: The strength-size effect of the glass ceramic material OTM 357 calculated from relative volumes $V/V_o$ and surface areas $S/S_o$ ($\sigma_o$ = 125 MPa) for the specimen 4×4×50 mm in size

It should be noted that under ceramic component manufacturing conditions it can be difficult to detect with assurance the surface flaws of this size by nondestructive testing methods. Thereby it is essential both to develop and to use the advanced nondestructive testing methods in the manufacturing of ceramic fairings and to use the probabilistic ratios based on brittle failure models of McClintock model type in order for the similar flaws and their effects to be evaluated.

The parameters of this model ***a*** and ***b*** can be determined from the median and from the ratio of quantiles of the empirical strength distribution. From equation (6) the ratio of the third and first quantiles is

$$\frac{s\,(0.75)}{s\,(0.25)} = r = \sqrt{\frac{\ln \dfrac{Aa/S}{-\ln 0.75}}{\ln \dfrac{Aa/S}{-\ln 0.25}}}\,,$$

wherefrom

$$\frac{Aa}{S} = \ln 4 \left( \frac{\ln 4}{\ln(4/3)} \right)^{1/(r^2-1)}. \qquad \textbf{(7)}$$

At the same time it follows from the function of the Weibull distribution[6] that

$$r = \frac{s\,(0.75)}{s\,(0.25)} = \left( \frac{\ln 0.25}{\ln 0.75} \right)^{1/m} \approx 4.82^{1/m}. \qquad \textbf{(8)}$$

The interquantile ratios obtained from the results of three-point flexure and direct tension tests of OTM 357 specimens of different size are in satisfactory agreement with relationship (8).

The value of the surface area $S$ depends on the size of material microstructure element $d$ which in its turn determines the flaw (crack) size $c$, that is $c \approx d$. Then the relationship of $S$ on the probability $w$ of the existence of the crack of size $c$ can be expressed as

$$S = d^2 (1/w)^{1/2}. \qquad \textbf{(9)}$$

Substituting equations (8), (9) and the expression for $a$ from equation (5) into equation (7), we obtain

$$\frac{w}{0.5 - 1/\ln w} = \frac{\ln 4 \left( \frac{\ln 4}{\ln(4/3)} \right)^{1/(4.82^{2/m} - 1)}}{A/d^2}. \qquad \textbf{(9)}$$

Now knowing the Weibull modulus $m$, the component surface area $A$ and the flaw size $d$, we can determine the probability $w$.

Fig. 3 shows the functions $A/S = f(m)$ obtained for the stressed surface area of the fairing shell $A = S_{ef} = 40{,}000$ mm$^2$ and $S$ calculated by equation (9) for different sizes of the material microstructure elements $d$. As can be seen from this figure the ratio $A/S$ satisfies the McClintock model.

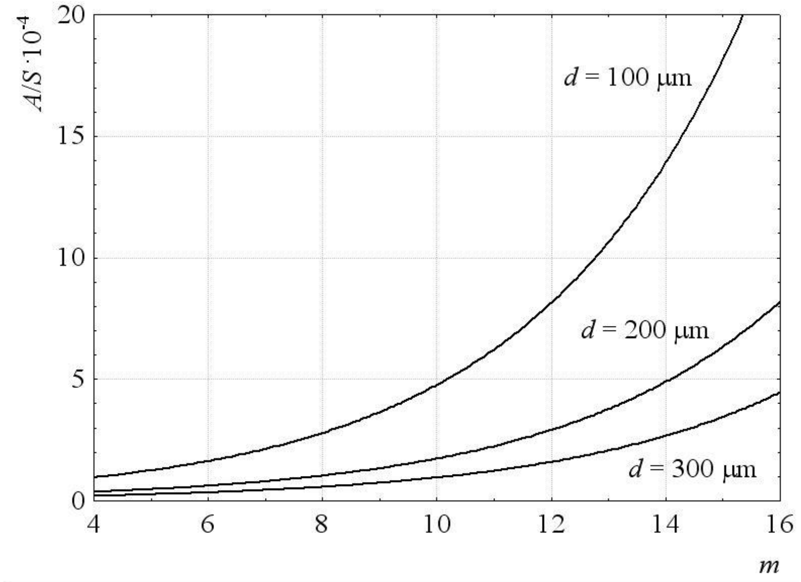


Figure 3: The dependence of the ratio of the component surface area to the microstructure element surface area on the Weibull modulus

The safety factor η for brittle ceramic materials representing the ratio of the average (median) material strength to the maximum stress $\sigma_{\max}$ acting on the component must be evaluated with due regard for the acceptable level of failure probability $P_f$, that is

$$h = \frac{s(P_f)}{s_{\max}} = k \frac{s(0.5)}{s_{\max}}, \qquad \textbf{(10)}$$

where the factor $k = \sigma(P_f)/\sigma(0.5)$ corrects the safety factor value in accordance with the required reliability of the component material. According to equation (6) the coefficient $k$ is

$$k = \left( \ln \frac{Aa/S}{-\ln(0.5)} \Big/ \ln \frac{Aa/S}{-\ln(1 - P_f)} \right)^{1/2}. \qquad \textbf{(11)}$$

Fig. 4 gives obtained by eq. (11) values of the correcting factor $k$ for a given value $A$ = 40,000 mm$^2$ versus the Weibull modulus, the size of material microstructure element $d$ and the failure probability $P_f$. It is obvious that the value of the coefficient $k$ and the safety factor decrease with the increase of the required reliability of the material, the size of the microstructure element (flaw) and with the deterioration of structure homogeneity (defect structure).

For the abovementioned effective sizes of the fairing from the glass ceramic material OTM 357 having the Weibull modulus $m$ = 5.5 which was obtained from the results of the material specimen uniaxial tension tests, the correcting factor $k$ at the failure probability $P_f$ = 0.01 and the flaw size $d$ = 200 is equal to 0.71. Then in case of the average uniaxial tensile strength of the specimens from OTM 357 equal to 68 MPa the safety factor of the fairing evaluated by eq. (9) is $\eta = ks(0.5)/s_{\max}$ = 0.71×68/18 = 2.7.

This result is close to the safety factor value evaluated above with the Weibull model and obtained in the field tests of the component. The advantage of the McClintock statistical approach consists in the possibility to predict the influence of the ratio of the component size to the microstructure flaw size on the reliability.

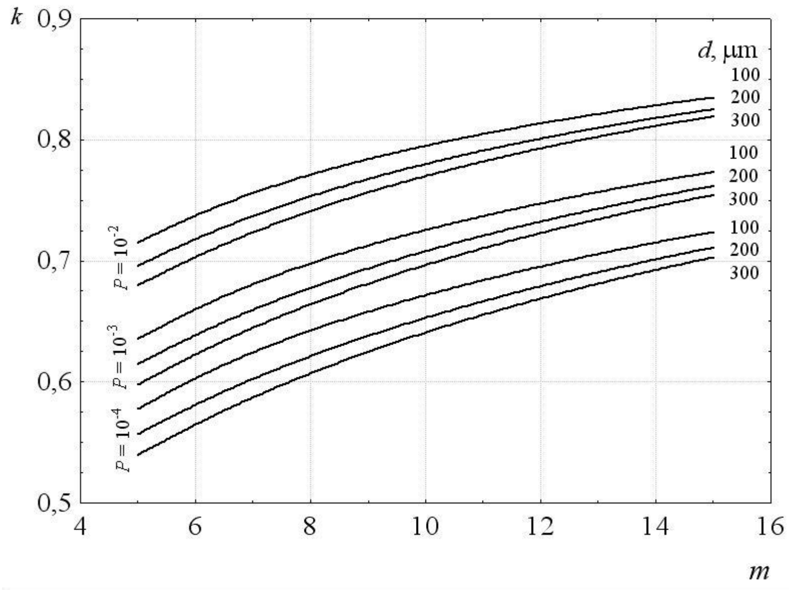


Figure 4: The correcting factor **k** versus the Weibull modulus, material structure element size and failure probability

Based on the results of the uniaxial tensile tests of glass ceramic specimens and on the value of critical stress intensity factor $K_{IC}$ = 1.33 MPa·m$^{1/2}$ the dependence of the density of flaws in the material on their size (Fig. 5) has been obtained.

The flaw density δ was calculated according to the following equation:

$$d_i = \frac{M_i}{N_i V} = \frac{M_i}{(N_{i-1} - M_{i-1})V}, i = 1,..., N, \quad \mathbf{(12)}$$

where $M_i$ is the number of failed specimens under i-th failure stress, $N_{i\text{-}1}$ is the number of specimens remained under lower stress after the exclusion of the failed specimens, $V$ is the effective volume of the uniaxial tensile test specimen, $N$ is the number of tested specimens.

The results allow us to conclude that about 2/3 out of all irregularities in the unit volume of the OTM 357 material are the flaws no more than 150 μm in size and 1/3 – from 150 to 300 μm and higher. The flaws higher than 300 μm are rare in occurrence (6 out of 100 specimens).

In accordance with the Weibull modulus estimation obtained by the maximum-likelihood technique the exponent $n$ in the function of flaw size distribution (1) is equal to 1.85. The parameter $k_c$ can be determined by fitting it to the empirical points (Fig. 5) of differential function of the flaw size distribution density

$$g(c) = k_c n c^{-n+1} = g_0 c^{-r}, \quad \mathbf{(13)}$$

resulting from its integral expression (1) where $g_o$ = $k_c n$, $r$ = $n$ + 1. Fitting by the least-square method makes it possible to obtain the function of the distribution density $g(c)$ = 152 $c^{-2.85}$.

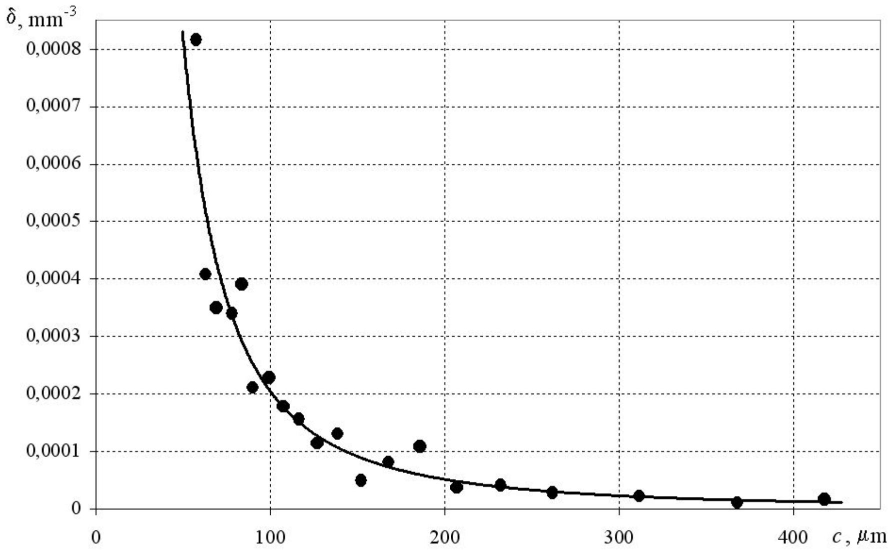


Figure 5: The dependence of flaw density on the flaw size

The probability that the flaws in the volume $V$ under uniform tension are larger than $c_1$ is

$$P(c \geq c_1) = 1 - \exp\left[ -V \int_{c_1}^{\infty} g(c) dc \right]. \qquad \textbf{(14)}$$

Using $g(c)$ obtained earlier, the probabilities of existence of flaws $\geq c_1$ have been calculated by eq. (14). These probabilities have been obtained both for a standard test specimen and for the glass ceramic shells with the symbols “component A1” and “component A2” which differ in volume by a factor of five (Fig. 6). It is obvious that in case of the probability equal to 1 there are the flaws 200 μm in size in the stressed volume of A1 shell while in the shell of the component A2 under the same probability there may be the flaws 400 μm in size. The probability of the flaws 400 μm in size in A1 is equal to 0.53.

To change from the flaw size distribution to the failure probability, let us insert eq. (13) into eq. (14), express the critical flaw size in terms of the maximum stress $\sigma_{\max}$ acting on the component and obtain the expression for the failure probability under the stress less than $\sigma_{\max}$

$$P_f = 1 - \exp\left[ -V \frac{g_0}{r-1} \left( \frac{\sigma_{\max} \sqrt{p}}{K_{IC}} \right)^{2(r-1)} \right]. \qquad \textbf{(15)}$$

The results of failure probability calculations by eq. (15) are given in Table 1. The safety factor was calculated as the ratio of the average (equal to 68 MPa) strength value of the specimens under tension to a maximum load.

Taking into account maximum stresses in A1 and A2, the flaws 400 μm are not critical as the failure stress equal to 40 MPa in this case is higher than the maximum performance stress. However the strength reliability of the component A2 having larger volume decreases as there may exist flaws of larger size in it. This fact should be taken into account in structural design.

Table 1: The probability of failure material of the components

| **Component** | **$\sigma_{\max}$, MPa** | **Safety factor** | **Stressed volume, mm$^3$** | **Critical flaw size, mm** | **Failure probability** |
|---|---|---|---|---|---|
| A1 | 18 | 3.7 | 130,000 | ≈3.6 | 0.0097 |
| A2 | 25 | 2.7 | 702,000 | ≈1.9 | 0.1630 |

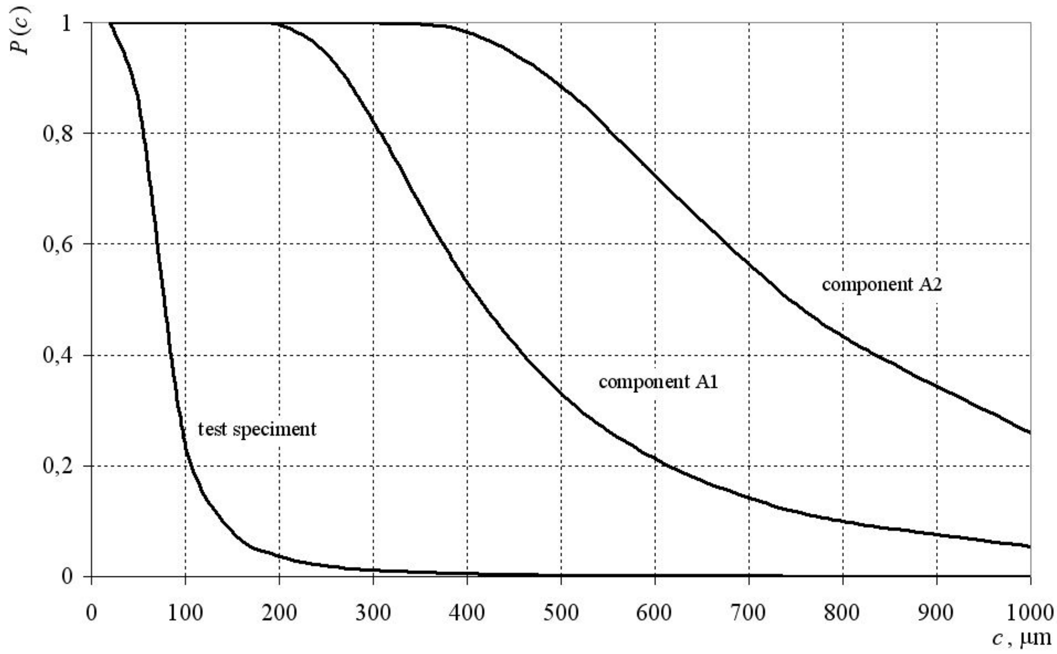


Figure 6: The probability of presence the flaws more than *c* in glass ceramic components

## 4. Conclusions

It is shown in the paper that the use of two different approaches based on the statistical data on strength and fracture mechanics relationships for the prediction of failure probability and safety factor of glass ceramic rocket fairings gives closed results. The possibility is shown to relate the integrity of the component material to a typical size of microstructure element or flaw and to predict the influence of dimensional relationship of the ceramic component and ceramic material microstructure element on the strength. It is also shown in the paper that in case of availability of large-sized flaws in the component material volume the safety factor and consequently the strength reliability of the component decrease. The probability interpretation of the safety factor is presented. The results obtained supplement the concept of relationship between the strength of the glass ceramic material OTM 357 and the availability of flaws in its structure.

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