SUPERCRITICAL AND TRANSCRITICAL INTERACTION OF SUPERSONIC FLOW WITH BOUDARY LAYER

Neyland V.Ya., Sokolov L.A.

TsAGI, Zhukovsky

The asymptotic theory of the interaction between an inviscid flow and a boundary layer is an important part of viscous gas dynamics at high Reynolds numbers. It is based on the fundamental idea of L. Prandtl on the possibility of dividing the entire flow region into an inviscid flow and a thin boundary layer (L. Prandtl [1904]). This idea appeared in connection with an attempt to provide an explanation for the phenomenon of flow separation from the body surface. We note that Prandtl's idea turned out to be very fruitful not only in viscous flow dynamics but also in many other branches of applied mathematics. The original formulation of boundary layer theory includes an assumption that the problem can first be solved for the outer inviscid flow and then for the boundary layer at the previously determined pressure distribution. Later Prandtl [1939] noted the possibility of refining the solution by taking account for the displacing effect of the boundary layer on the outer flow. In the next approximation, the effect of outer flow variations on the boundary layer flow must be taken into account, etc. Actually, the concept of weak interaction theory was thus formulated.

However, further studies, both experimental and theoretical, showed that in many cases taking weak interaction into account does not solve all resulting problems and, in particular, does not enable one to describe the flow near the boundary layer separation point. An analysis of the solutions of the boundary layer equations at a given outer pressure distribution leads often to the appearance of an "impassable" singular point (L. D. Landau and E. M. Lifshitz [1944]; S. Goldstein [1948]). The key role played by strong local interaction between the outer flow and the boundary layer leading to the removal of the singularity of solutions and the possibility of continuing it through the separation point was detected in the works of V. Ya. Neiland [1968, 1969*a*] and K. Stewartson and P. G. Williams [1969]. In the vicinity of the laminar separation point in a supersonic flow the asymptotic representation of the solutions of the Navier-Stokes equations is of three-layer nature. Later, the use of the three-layer limiting structure of the solution turned out to be very fruitful.

Moreover, the asymptotic approach to the solution of problems of supersonic viscous flow dynamics made it possible to examine a wide class of problems, which are not necessarily reduced to the three-layer flow pattern and are not governed by the classical boundary layer theory. These include, for example, flow near the shock incidence onto a boundary layer, flows past corner points on bodies and various steps on the boundary layer bottom, and other flows including strong local interaction regions, as well as the problems in which the interaction is strong over the entire body surface, which can lead to upstream disturbance transfer when the body is in a hypersonic flow. Apparently, the most complete survey of the studies in these lines of inquiry for supersonic gas flows is presented in the paper of V. Ya. Neiland and etc.[2004].

Thus, the range of problems described on the basis of the search for limiting solutions of the Navier-Stokes equations at the passage to the limit $Re \to \infty$ has been expanded widely. Now it includes a large number of problems, important for both theory and practice, which cannot be described by the classical boundary layer theory. It is particularly important also for the reason that determining numerical solutions for high Re is difficult. Moreover, the asymptotic approach provides a rational explanation to the physical properties of the flows and helpful approximate similarity laws.

The structure of the supersonic flow/boundary layer interaction regions, the direction of disturbance transfer, and the scale of distances through which disturbances can propagate can be dependent on the Mach number profile in the undisturbed boundary layer. L. Crocco [1955] was the first who detected this fact. He developed the idea of subcritical and supercritical boundary layers, which are either capable of transferring pressure disturbances upstream through considerable distances, as compared with the boundary layer thickness, or do not possess this property. Crocco noted that, in order for a supercritical boundary layer to acquire this property, it is necessary that the boundary-layer Mach number profile restructures itself sharply, or, in accordance with Crocco's terminology, supercritical jump occurs. These ideas were substantiated on the basis of general physical considerations and the integral method for describing the interaction.

However, earlier studies on the asymptotic theory of separation and interaction preceding the work of V. Ya. Neiland [1973] considered actually only subcritical regimes. It was even suggested that the supercriticality property is a consequence of inaccurate description of the phenomenon when using the integral boundary-layer equations rather than a physical property of the flows (S. N. Brown and K. Stewartson [1969]). However, in the studies of V. Ya. Neiland [1973, 1974, 1987] an asymptotic theory of two- and three-dimensional supercritical flows was developed and a fundamental analogy between the properties of subsonic and supersonic inviscid gas flows, on the one hand, and subcritical and supercritical boundary layers in an outer supersonic flow, on the other hand, was established. In this case, the description of transcritical flows, similar to the transonic velocity regime in conventional gasdynamics, was of fundamental interest.

1 Strong interaction between a hypersonic flow and the boundary layer on a cold delta wing

We will consider the distinctive features of disturbance propagation and the formulation of the boundary value problem for the boundary layer on a delta wing in the case in which the interaction with the outer hypersonic flow is non-weak, while the body temperature is low as compared with the freestream stagnation temperature ($g_w \ll 1$). The results obtained make it possible to draw some interesting conclusions concerning the general three-dimensional flow.

Let us introduce a Cartesian coordinate system with origin at the wing vertex, the xl and zl axes directed normal to one of the wing edges and along this edge in the wing plane, respectively, and the $yl\tau$ axis normal to the wing plane (Fig. 1).

In accordance with the conventional estimates for the boundary layer in a hypersonic flow (W. D. Hayes and R. F. Probstein [1966]), we will use the notation

$$u_\infty u, \quad u_\infty w, \quad \tau u_\infty v, \quad \tau^2 \rho_\infty \rho$$

$$\tau^2 \rho_\infty u_\infty^2 p, \quad \left(u_\infty^2/2\right) g, \quad \mu_0 \mu$$

for the velocity components, the density, the pressure, the stagnation enthalpy, and viscosity. The parameter $\tau = Re_0^{-1/4}$, where the Reynolds number $Re_0 = \rho_\infty u_\infty \ell / \mu_0$ is based on the freestream density and velocity, the viscosity at the freestream stagnation temperature, and a scale length ℓ which in the self-similar case drops out of final results.

In the Dorodnitsyn variables the boundary layer equations and the boundary conditions take the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v^*}{\partial \eta} + \frac{\partial w}{\partial z} = 0, \qquad \frac{\partial p}{\partial \eta} = 0, \qquad v^* = \rho v + u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z} \tag{1}$$

$$u\frac{\partial u}{\partial x} + v^*\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial \eta}\left(\mu\rho\frac{\partial u}{\partial \eta}\right)$$

$$u\frac{\partial w}{\partial x} + v^*\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial}{\partial \eta}\left(\mu\rho\frac{\partial w}{\partial \eta}\right)$$

$$u\frac{\partial g}{\partial x} + v^*\frac{\partial g}{\partial \eta} + w\frac{\partial g}{\partial z} = \frac{\partial}{\partial \eta}\left\{\rho\mu\left[\frac{1}{\sigma}\frac{\partial g}{\partial \eta} + \left(1 - \frac{1}{\sigma}\right)\frac{\partial}{\partial \eta}(u^2 + w^2)\right]\right\}$$

$$p = \frac{\gamma - 1}{2\gamma}(g - u^2 - w^2)\rho, \qquad \eta = \int_0^y \rho\, dy, \qquad \delta = \int_0^\infty \frac{d\eta}{\rho}$$

$$\rho\mu = \frac{2\gamma}{\gamma - 1}p(g - u^2 - w^2)^{n-1}, \qquad p = \frac{\gamma + 1}{2}\left(\frac{\partial \delta}{\partial x}\sin\omega_0 + \frac{\partial \delta}{\partial z}\cos\omega_0\right)^2$$

$$u_w = v_w^* = w_w = g_w = 0, \qquad u_e = \sin\omega_0, \qquad w_e = \cos\omega_0, \qquad g_e = 1$$

Here, ω_0 is the angle between the z axis (wing edge) and the freestream direction.

For further analysis we will make the change of variables

$$(x,\ \eta,\ z) \to \left(x,\ \zeta = \frac{1}{\omega_0}\arctan\frac{x}{z},\ \lambda = \eta/x^{1/4}\right) \tag{2}$$

$$u = \frac{\partial \psi}{\partial \eta}, \qquad w = \frac{\partial \varphi}{\partial \eta}, \qquad v^* = -\left(\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial z}\right)$$

$$\psi = x^{1/4} f_*, \quad \varphi = x^{1/4}\varphi_*, \quad p = x^{-1/2} p_*, \quad \rho = x^{-1/2}\rho_*, \quad \delta = x^{3/4}\delta_*$$

We will write down the equations and the boundary conditions in the new variables omitting asterisks

$$(Nf'')' + \frac{1}{4}ff'' + \frac{\gamma - 1}{4\gamma}\left(1 - \frac{\sin 2\omega \cdot \dot{p}}{\omega_0 p}\right)(g - f'^2 - \varphi'^2) =$$
$$\frac{\sin 2\omega}{2\omega_0}(f'\dot{f}' - f''\dot{f}) - \frac{\sin^2\omega}{\omega_0}(\varphi'\dot{f}' - f''\dot{\varphi}) \tag{3}$$

$$(N\varphi'')' + \frac{1}{4}f\varphi'' + \frac{\gamma-1}{2\gamma}\frac{\sin^2\omega}{\omega_0}\frac{\dot{p}}{p}(g - f'^2 - \varphi'^2) =$$
$$\frac{\sin 2\omega}{2\omega_0}(f'\dot{\varphi}' - \varphi''\dot{f}) - \frac{\sin^2\omega}{\omega_0}(\varphi'\dot{\varphi}' - \varphi''\dot{\varphi})$$

$$\left(\frac{N}{\sigma}g'\right)' + \frac{1}{4}gf' + \left[N\left(1 - \frac{1}{\sigma}\right)\left(f'^2 + \varphi'^2\right)'\right]' =$$
$$\frac{\sin 2\omega}{2\omega_0}(f'\dot{g} - g'\dot{f}) - \frac{\sin^2\omega}{\omega_0}(\varphi'\dot{g} - g'\dot{\varphi})$$

$$N = \frac{2\gamma}{\gamma-1}p\left(g - f'^2 - \varphi'^2\right)^{n-1}, \quad \omega = \omega_0\zeta$$

$$\delta = \frac{\gamma-1}{2\gamma p}\int\limits_0^\infty \left(g - f'^2 - \varphi'^2\right)\,d\lambda, \quad F = \int\limits_0^\infty \left(g - f'^2 - \varphi'^2\right)\,d\lambda$$

$$f_w = \varphi_w = f'_w = \varphi'_w = g_w = 0, \quad f'_e = \sin\omega_0, \quad \varphi'_e = \cos\omega_0, \quad g_e = 1$$

$$p = \frac{\gamma+1}{2}\left[\frac{3}{4}\delta\sin\omega_0 + \frac{1}{\omega_0}\frac{d\delta}{d\zeta}\left(\frac{1}{2}\sin 2\omega\sin\omega_0 - \sin^2\omega\cos\omega_0\right)\right]^2$$

1.1 Solution near the leading edge

As usually (V. Ya. Neiland [1970*b*]; I. G. Kozlova and V. V. Mikhailov [1970]), we can construct an expansion for the solution near the leading edge

$$p(\zeta) = p_0 + p_1\zeta^a + \cdots, \quad f(\zeta,\lambda) = f_0(\lambda) + \frac{p_1}{p_0}\zeta^a f_1(\lambda) + \cdots \tag{4}$$
$$\varphi(\zeta,\lambda) = \varphi_0(\lambda) + \frac{p_1}{p_0}\zeta^a\varphi_1(\lambda) + \cdots, \quad g(\zeta,\lambda) = g_0(\lambda) + \frac{p_1}{p_0}\zeta^a g_1(\lambda) + \cdots$$
$$N(\zeta,\lambda) = N_0(\lambda) + \frac{p_1}{p_0}\zeta^a N_1(\lambda) + \cdots, \quad \delta(\zeta) = \delta_0 + \frac{p_1}{p_0}\zeta^a\delta_1 + \cdots$$
$$F(\zeta) = F_0 + \frac{p_1}{p_0}\zeta^a F_1 + \cdots$$

An equation for the zeroth approximation can be derived from Eq. (3) by letting $\omega = 0$ in this equation. The next terms of the expansion are determined from the system of linear equations

$$
\begin{gathered}
(N_1 f_0'' + N_0 f_1'')' + \frac{1}{4}(f_0 f_1'' + f_1 f_0'') + \\
\frac{\gamma - 1}{4\gamma}\left[g_1 - 2f_0' f_1' - 2\varphi_0'\varphi_1' - 2a\left(g_0 - f_0'^2 - \varphi_0'^2\right)\right] = a(f_0' f_1' - f_0'' f_1)
\end{gathered} \tag{5}
$$

$$
(N_1 \varphi_0'' + N_0 \varphi_1'')' + \frac{1}{4}(f_0 \varphi_1'' + f_1 \varphi_0'') = a(f_0' \varphi_1' - \varphi_0'' f_1)
$$

$$
\begin{gathered}
\left(\frac{N_0}{\sigma} g_1' + \frac{N_1}{\sigma} g_0'\right)' + \frac{1}{4}(f_0 g_1' + f_1 g_0') + \\
\left\{\left(1 - \frac{1}{\sigma}\right)\left[N_1\left(f_0'^2 + \varphi_0'^2\right)' + 2N_0(f_0' f_1' + \varphi_0' \varphi_2')'\right]\right\}' = a(f_0' g_1 - g_0' f_1)
\end{gathered}
$$

$$
N_1 = N_0 \left[1 - \frac{(n-1)(g_1 - 2f_1' f_0' - 2\varphi_1' \varphi_0')}{g_0 - f_0'^2 - \varphi_0' 2}\right]
$$

The boundary conditions for system (5) are trivial. After substitution of Eq. (4) in formulas for δ and p in Eq. (3), they give two independent relations determining δ_1. The constant p_1 drops out from the consideration owing to the special form of expansion (4) involving p_1 in the form of a factor of all functions. If p_1 is introduced only in the expansion for the function $p(\zeta)$, then we obtain a system of linear homogeneous equations resolvable only for an eigenvalue a determined from the same condition as for Eq. (5), namely

$$
a = \frac{3}{4}\left[\frac{F_0}{2(F_1 - F_0)} - 1\right] \tag{6}
$$

Of course, solution (4) – (6) is determined correct to the arbitrary constant p_1, if a nontrivial solution of the problem exists. However, numerical investigations showed that for $g_w = 0$ eigensolutions exist only for wings with a large sweep of the leading edge (small values of ω_0). The dependence of the eigenvalue a on the sweep angle $\frac{\pi}{2} - \omega_0$ is plotted in Fig. 2 for $\sigma = n = 1$. Thus, for wings with fairly large values of ω_0 there exists a region adjoining the leading edge within which a self-similar flow, which coincides with the flow past a semi-infinite swept plate, is realized, as it was suggested by M. D. Ladyzhenskii [1965], though for almost the whole wing. An analogous phenomenon of disturbance blocking on cold bodies in supersonic flows with free interaction was found by V. Ya. Neiland [1973]. The extent of the supercritical flow region ω_1 is presented in Fig. 2.

1.2 Flow regimes

The nature of the flow over a cold delta wing depends on the regime of the boundary layer/outer hypersonic flow interaction (V. Ya. Neiland [1974]). In m-subcritical regimes disturbances can propagate from the plane of symmetry toward the edges. For choosing an unique solution an additional boundary condition should be taken, for example, the value of the pressure at a certain surface ζ = const. In the supercritical flow region disturbances do not propagate upstream and the flow is governed by the self-similar solution of system (3). On a cold wing displacement thickness variation is produced by the main part of the boundary layer and this variation depends linearly on the pressure disturbance

(V. Ya. Neiland [1974]). The flow regime is determined by the sign of the derivative $\frac{d\delta}{dp}$. The supercritical regime is realized for $\frac{d\delta}{dp} < 0$ and the subcritical regime for $\frac{d\delta}{dp} > 0$.

With decrease in the wing sweep angle $\chi = \frac{\pi}{2} - \omega_0$ the derivative $\left.\frac{d\delta}{dp}\right|_{\zeta=0}$ decreases monotonically, vanishes at a certain critical value $\chi^* = \frac{\pi}{2} - \omega^*$, and remains negative with further decrease in the sweep angle to zero. Thus, for $\omega_0 > \omega^*$ in the delta-wing boundary layer there exist the regions of both supercritical and m-subcritical flow and transition from the flow of one type to the other takes place at a certain value $\omega^* < \omega_1 < \omega_0$ (V. Ya. Neiland [1974]). The flow in the region between the leading edge and the $\omega = \omega_1$ surface is governed by the self-similar solution of system (3). In the region between the $\omega = \omega_1$ surface and the plane of symmetry of the wing the effect of the upstream disturbance transfer should be taken into account.

We note that in proving the principle of impossibility of disturbance localization in the flows with non-weak interaction between the boundary layer and the outer hypersonic flow (V. Ya. Neiland [1970*b*]) the flow pattern in a thin wall layer, in which velocity disturbances induced by a small pressure difference are of the same order as the velocity itself, is important. Precisely a large value of $\Delta\delta \sim g_w \Delta p^{1/2}$ produced by this sublayer at small pressure disturbances Δp leads to the conclusion on inevitable upstream disturbance transfer through finite distances. However, for $g_w \to 0$ (one more case of disturbance blocking under the action of an intense expansion wave was studied by V. Ya. Neiland [1972]) the quantity $\Delta\delta$ depends completely on the deformation of the flow parameter profiles in the main part of the boundary layer. There $\Delta\delta \sim \Delta p$, while the sign of $d\delta/dp$ depends on the Mach number profile across the boundary layer. If $d\delta/dp < 0$, then, in accordance with the terminology introduced for plane flows by L. Crocco [1955], the layer is supercritical and the disturbance transfer is absent. The subcritical case $d\delta/dp > 0$ corresponds to upstream disturbance propagation. The rough integral approach adopted in the work of L. Crocco [1955] did not take account for the effect of sublayers and led to a qualitative discrepancy with rigorous solutions when studying the flows with $g_w \sim O(1)$. However, for the flows with $g_w \to 0$ there appears a qualitative agreement: the flow, "subsonic" in the mean, $(d\delta/dp > 0)$ transmits disturbances, whereas the "supersonic" flow does not.

On transition to three-dimensional flows the analogy with the subsonic and supersonic types of the pressure disturbance transfer is also conserved. The solution of I. G. Kozlova and V. V. Mikhailov [1970] with $g_w = O(1)$ corresponds to the subsonic case, while the above solution with $g_w = 0$ to the supersonic case. However, in an inviscid supersonic flow (we note that even for $g_w = 0$ the characteristic value of the Mach number within the boundary layer is not hypersonic, $M \sim 1$) within Mach cones there are directions, along which disturbances are transferred. For this reason, we will consider a delta wing with a large ω_0 and $g_w = 0$ and seek a value of ω_1 at which the eigensolutions of the problem, which make it possible to take account for disturbances proceeding from the plane of symmetry or neighboring regions with large ζ, appear at a certain ray $0 < \zeta_1 < 1$, unknown beforehand.

In the region $0 < \zeta < \zeta_1$ there is a self-similar solution corresponding to the flow over a flat plate with a leading edge which is not necessarily perpendicular to the freestream velocity.

For $\lambda = O(1)$ the viscosity forces have no effect on the disturbed boundary layer flow near the $\omega = \omega_1$ plane. An analysis of the system of equations governing this flow shows (G. N. Dudin and I. I. Lipatov [1985]) that it can be once integrated with respect to ζ; hence follows that the disturbances of the functions f and g are proportional to the pressure disturbance $p_1(\zeta_1)$. To the right of the $\omega = \omega_1$ plane, we will represent the solution in the form of the expansions for the flow functions

$$p = p_0 + p_1(\zeta_1) + \cdots, \quad \delta = \delta_0 + \delta_1 + \cdots, \quad \zeta_1 = \zeta - \frac{\omega_1}{\omega_0} \tag{7}$$

$$f = f_0(\lambda) + \frac{p_1(\zeta_1)}{p_0} f_1(\lambda) + \cdots, \quad \varphi = \varphi_0(\lambda) + \frac{p_1(\zeta_1)}{p_0}\varphi_1(\lambda) + \cdots$$

$$g = g_0(\lambda) + \frac{p_1(\zeta_1)}{p_0} g_1(\lambda) + \cdots$$

where functions with the subscript 0 correspond to the self-similar solution. Substituting Eq. (7) in the system of equations (3) yields a linear system of equations for the first approximation, the solution of which was derived by V. Ya. Neiland [1974]. We note that for $\lambda = O(1)$ this solution is independent of the form of the function $p_1(\zeta_1)$.

The equations for zeroth terms correspond to Eq. (3) for $\omega = 0$ (self-similarity in the undisturbed flow region)

$$-\frac{\gamma-1}{4\gamma}\sin 2\omega_1 \left(g_0 - f_0'^2 - \varphi_0'^2\right) = \frac{\sin 2\omega_1}{2}(f_0' f_1' - f_0'' f_1) - \sin^2\omega_1(\varphi_0' f_1' - f_0''\varphi_1) \tag{8}$$

$$\frac{\gamma-1}{2\gamma}\sin^2\omega_1 \left(g_0 - f_0'^2 - \varphi_0'^2\right) = \frac{\sin 2\omega_1}{2}(f_0'\varphi_1' - \varphi_0'' f_1) - \sin^2\omega_1(\varphi_0'\varphi_1' - \varphi_0''\varphi_1)$$

$$\frac{1}{2}\sin 2\omega_1(f_0' g_1 - g_0' f_1) - \sin^2\omega_1(\varphi_0' g_1 - g_0'\varphi_1) = 0$$

$$\omega_1 = \omega_0\zeta_1, \quad (\delta_1 + \delta_0)\frac{2\gamma}{\gamma-1}p_0 = F_1$$

$$F_1 = \int_0^\infty (g_1 - 2f_1' f_0' - 2\varphi_1'\varphi_0')\, d\lambda, \quad \delta_1 = 0$$

These expansions are invalid in the vicinity $\lambda \to 0$, where it is necessary to consider a region $\overline{\lambda} = \lambda/\sigma(\Delta\zeta) = O(1)$ with the scale $\sigma(\Delta\zeta) \to 0$ as $\Delta\zeta \to 0$, while $\sigma(\Delta\zeta)$ must be so introduced that the viscous terms of Eqs. (3) are retained and the solution admits matching with solution (8).

Equation (8) can be integrated as follows:

$$f_1' = \frac{\gamma-1}{2\gamma}\left[-\frac{\cos\omega_1\left(g_0 - f_0'^2 - \varphi_0'^2\right)}{\cos\omega_1 f_0' - \sin\omega_1\varphi_0'} - f_0''\int_0^\lambda \frac{g_0 - f_0'^2 - \varphi_0'^2}{(\cos\omega_1 f_0' - \sin\omega_1\varphi_0')^2}\, d\lambda\right] \tag{9}$$

$$\varphi_1' = \frac{\gamma-1}{2\gamma}\left[\frac{\sin\omega_1\left(g_0 - f_0'^2 - \varphi_0'^2\right)}{\cos\omega_1 f_0' - \sin\omega_1\varphi_0'} - \varphi_0''\int_0^\lambda \frac{g_0 - f_0'^2 - \varphi_0'^2}{(\cos\omega_1 f_0' - \sin\omega_1\varphi_0')^2}\, d\lambda\right]$$

$$g_1 = -\frac{\gamma-1}{2\gamma}g_0'\int_0^\eta \frac{g_0 - f_0'^2 - \varphi_0'^2}{(\cos\omega_1 f_0' - \sin\omega_1\varphi_0')^2}\, d\lambda$$

According to the last formulas (8), in order for an eigensolution to exist the quantity ζ_1 must satisfy the equation

$$F_0 = \frac{\gamma-1}{2}\int\limits_0^\infty \left(\frac{g_0 - {f_0'}^2 - {\varphi_0'}^2}{\cos\omega_1 f_0' - \sin\omega_1 \varphi_0'}\right)^2 d\lambda \tag{10}$$

where

$$F_0 = \int\limits_0^\infty \left(g_0 - {f_0'}^2 - {\varphi_0'}^2\right) d\lambda$$

Then in region $0 \le \zeta \le \zeta_1$ the solution is self-similar, while the disturbed part of the flow region and the eigensolutions determined correct to an arbitrary constant p_1 appear for $\zeta = \zeta_1$. The dependence of ω_1 on the leading edge sweep angle $\frac{\pi}{2} - \omega_0$ is presented in Fig. 2.

The expression for the displacement thickness formed in the $\lambda = O(1)$ region is as follows:

$$\delta_1 = \frac{\gamma-1}{2\gamma p_0}\Delta p\left[\frac{\gamma-1}{2}\int\limits_0^\infty \left(\frac{g_0 - {f_0'}^2 - {\varphi_0'}^2}{f_0'\cos\omega_1 - \varphi_0'\sin\omega_1}\right)^2 d\lambda - \int\limits_0^\infty \left(g_0 - {f_0'}^2 - {\varphi_0'}^2\right) d\lambda\right] \tag{11}$$

where $\Delta p = \frac{p_1}{p_0}$.

The derivative $\frac{d\delta_1}{d\Delta p}$ is linearly dependent on the coordinate ζ. Differentiating Eq. (11) with respect to the coordinate ζ gives the following expression

$$\frac{d\delta_1}{d\Delta p} = \frac{(\gamma-1)^2}{4\gamma p_0}\omega_0 J_1 \zeta_1, \quad J_1 = 2\int\limits_0^\infty \left(g_0 - {f_0'}^2 - {\varphi_0'}^2\right)\frac{f_0'\sin\omega_1 + \varphi_0'\cos\omega_1}{(f_0'\cos\omega_1 - \varphi_0'\sin\omega_1)^3}\, d\lambda \tag{12}$$

The solution for the first approximation is not uniformly accurate, since it does not take the viscosity effect into account. In order to satisfy the boundary conditions imposed on the wing surface, it is necessary to bring into consideration region 2 in which the influence of the viscosity and inertia forces is the same in the first approximation. The wall region 2 (Fig. 1) induces a variation of the displacement thickness Δ_1. The estimates for the region 2 thickness and the scales of the functions in this region are obtained by equating the orders of the terms of the system of equations responsible for the effects of the viscosity and inertia forces. Finally, the requirement of the matching of solutions in regions 1 and 2 leads to the equality of the orders of the pressure gradient and the inertial terms thus making it possible to determine – at the known region 2 thickness – the scales of the disturbed functions f, φ, and g. The expression for the variation of the displacement thickness formed in region 2 is as follows:

$$\Delta_1 = c_1 p_1 \left(\frac{p_1}{\dot p_1}\right)^{1/3}, \qquad c_1 = \mathrm{const} \tag{13}$$

For determining the function $p_1(\zeta_1)$ the interaction condition must be used. The assumption that the main contribution to the displacement thickness variation is formed in region 1 leads, in view of the interaction condition, to a function of the form $p_1 = c\zeta_1^\alpha$. This expression can be fairly simply obtained by substituting δ_1 (11) in the expression for $p_1 = c\zeta_1^\alpha$

obtained by substitution of the expansions for the flow functions (7) in the expression for the pressure (see Eq. (3)) with account for Eq. (12). Then the flow in region 2 induces the thickness variation $\Delta_1 \approx \zeta_1^{1/3} p_1$ which is greater in the order than $\delta_1 \approx \zeta_1 p_1$ for all permissible values $\alpha > 0$. The assumption that the main contribution to the displacement thickness is formed in region 2 also does not lead to a self-consistent scheme. In this case it turns out that the induced pressure disturbance is greater than the original disturbance.

An analysis of the interaction condition shows that a solution of Eq. (7) exists if in the first approximation the total variation of the displacement thickness is equal to zero

$$\Delta\delta = \delta_1 + \Delta_1 = 0 \tag{14}$$

The interaction of analogous nature was described in the papers of K. Stewartson [1969], in which the flow in the vicinity of the leading edge of a plate was studied, and V. V. Bogolepov and V. Ya. Neiland [1976], in which the flow over small roughnesses at the bottom of a laminar boundary layer was investigated. The expression for the function $p_1(\zeta_1)$ follows from Eqs. (12) and (13) and takes the form:

$$p_1(\zeta_1) = c\exp\left(-\frac{\alpha}{\zeta_1^2}\right) \tag{15}$$

The total displacement thickness variation can be derived form Eq. (3)

$$\Delta\delta = \frac{\gamma-1}{2\gamma p_0}\Delta p\left[\int_0^\infty \left(g_1 - 2f_0'f_1' - 2\varphi_0'\varphi_1' - g_0 + f_0'^2 + \varphi_0'^2\right)d\lambda + \zeta_1\int_0^\infty (g_2 - g_1(0))\,d\eta\right] + O\left(\zeta_1^2\frac{p_1}{p_0}\right), \quad \eta = \frac{\lambda}{\sqrt{\zeta_1}} \tag{16}$$

The first integral on the right side of Eq. (16) represents the contribution of region 1 and, in accordance with Eq. (12), can be written in the form $\delta_1 = \frac{(\gamma-1)^2}{4\gamma p_0}\Delta p\,\omega_0 J_1\zeta_1$. The second integral in Eq. (16) is the contribution of the wall region to the displacement thickness variation; for determining this integral a solution for region 2 should preliminarily be found.

In region 2 we introduce the following expansions of the functions

$$f = \frac{1}{2}a\eta^2\zeta_1 + \frac{c}{p_0}\sqrt{\zeta_1}\exp\left(-\frac{\alpha}{\zeta_1^2}\right)f_2(\eta) + \cdots \tag{17}$$

$$\varphi = \frac{1}{2}b\eta^2\zeta_1 + \frac{c}{p_0}\sqrt{\zeta_1}\exp\left(-\frac{\alpha}{\zeta_1^2}\right)\varphi_2(\eta) + \cdots$$

$$g = d\eta^2\zeta_1^{1/2} + \frac{c}{p_0}\exp\left(-\frac{\alpha}{\zeta_1^2}\right)g_2(\eta) + \cdots$$

$$p = p_0 + c\exp\left(-\frac{\alpha}{\zeta_1^2}\right) + \cdots, \quad \eta = O(1)$$

where c is an arbitrary constant. The first terms of the expansions for the functions f, φ, and g are the asymptotic representations of the functions f_0, φ_0, and g_0 as $\lambda \to 0$. The parameters a, b, and d are determined from the self-similar solution. Substituting Eq. (17) in the system of equations (13) leads after certain transformations to the following first-approximation system of equations

$$z_1''' - \eta_1 = \eta_1 z_1', \quad z_2''' - \eta_1 = \eta_1 z_2' - z_2, \quad z_3''' - \eta_1 = \eta_1 z_3 - z_3$$

$$z_1(0) = z_1'(0) = z_2(0) = z_2'(0) = z_3(0) = 0$$

$$z_1'(\infty) = -1 + O(1), \quad z_2'(\infty) = -\ln \eta_1 + O(1), \quad z_3'(\infty) = -\ln \eta_1 + O(1)$$

where

$$z_1 = 2^{4/3}\alpha^{1/3}\gamma\omega_0 T^{4/3}\frac{bf_2 - a\varphi_2}{d(\gamma-1)\sin\omega_1(b\cos\omega_1 + a\sin\omega_1)}$$

$$z_2 = 2^{4/3}\alpha^{1/3}\gamma\omega_0 T^{4/3}\frac{f_2\cos\omega_1 - \varphi_2\sin\omega_1}{d(\gamma-1)\sin\omega_1}$$

$$z_3 = 2\gamma\omega_0^2 T^2\frac{g_2}{d(\gamma-1)\sin^2\omega_1}$$

$$\eta = \eta_1 T^{-1/3} 2^{-1/3}\alpha^{-1/3}, \quad T = \frac{\sin\omega_1}{\omega_0(a\cos\omega_1 - b\sin\omega_1)}$$

The solution of the system of equations governing the flow in region 2 leads, in view of Eqs. (14) and (16), to an expression for the eigenvalue α

$$\alpha = \frac{1}{2}\left(\frac{d^2 J_2 \sin^2\omega_1}{\gamma\omega_0^3}\right)^3, \quad J_2 = \int_0^\infty (z_3(\eta_1) - z_3(\infty))\, d\eta$$

The dependence $\alpha(\chi)$ is presented in Fig. 3. Clearly, with increase in the leading edge sweep angle $\chi = \frac{\pi}{2} - \omega_0$ the eigenvalue $\alpha(\chi)$ increases monotonically and tends to infinity as the sweep angle tends to the value $\chi \to \chi^* = \frac{\pi}{2} - \omega^*$. The increase of the eigenvalue $\alpha(\chi)$ is due to a decrease in the intensity of the upstream disturbance transfer. This variation of the nature of the disturbance transfer can be attributed to the fact that in region 2 the thickness displacement variation is inverse proportional to the exponents of both the eigenvalue $\alpha(\chi)$ and the parameter ω_1.

Condition (14) requires the conservation of the quantity Δ_1; therefore, with decrease of the distance from the leading edge to the plane, in which the disturbed flow begins, the eigenvalue $\alpha(\chi)$ must increase.

The physical meaning of the growth of the disturbance transfer intensity (decrease of $\alpha(\chi)$) with decrease in the wing sweep is related with the fact that the supercritical flow region length increases (in ζ) with increasing ω_0; therefore, the flow function gradients decrease, the velocity profiles become less convex, and the thickness of the layer formed by subsonic streamtubes increases.

As $\omega_0 \to \omega^*$, the functions $\sin^2\omega$ and $\sin 2\omega$ entering in the system of equations (3) can be approximated by the first terms of the series expansion, since $\omega = \omega_1 + \omega_0 = O(1)$. Then for estimating the region 2 thickness we obtain

$$\lambda_2 \sim \left[\frac{p_1}{(\omega_1 + \omega_0\zeta_1)\dot{p}_1}\right]^{1/3}$$

Accordingly, the displacement thickness variation formed in region 2 is determined as follows:

$$\Delta_1 = Ap_1 \left[\frac{p_1}{(\omega_1 + \omega_0 \zeta_1)\dot{p}_1} \right]^{1/3}, \quad A = \left(\frac{\gamma - 1}{2\gamma} \right)^2 \frac{d^2 \omega_0^{1/3} J_2}{p_0 a^{7/3}} < 0 \tag{18}$$

In view of the expression for Δ_1, condition (14) takes the form:

$$B\zeta_1 p_1 + Ap_1 \left[\frac{p_1}{(\omega_1 + \omega_0 \zeta_1)\dot{p}_1} \right]^{1/3}, \quad B = \frac{(\gamma - 1)^2}{4\gamma p_0} \omega_0 J_1 \tag{19}$$

The solution of the differential equation (19) is as follows:

$$p_1 = c \exp \left\{ \frac{A^3}{\omega_0 B^3} \left(\frac{\omega_0}{\omega_1} \right)^3 \left[\ln \left(\frac{\omega_0 \zeta_1}{\omega_1 + \omega_0 \zeta_1} \right) + \frac{\omega_1}{\omega_0 \zeta_1} - \frac{\omega_1^2}{2\omega_0^2 \zeta_1^2} \right] \right\} \tag{20}$$

For $\omega_1 = O(1)$ expression (19) coincides in the leading term with Eq. (15). As $\omega_1 \to 0$ and for a small but fixed value ζ_1, the right side of Eq. (20) having been expanded in a series in the small parameter $\frac{\omega_1}{\omega_0 \zeta_1}$ leads to the function

$$p_1 = c \exp \left(\frac{A^3}{3B^3 \omega_0 \zeta_1^3} \right) \tag{21}$$

As $\omega_0 \to \omega^*$ $(\omega_0 - \omega^* < 0)$ the variation of the displacement thickness of region 1 can be presented in the form $\delta_1 = B\left(\zeta_1 - 1 + \frac{\omega^*}{\omega_0}\right) p_1$. The estimate for the region 2 thickness is obtained from Eq. (18) letting there $\omega_1 = 0$. Accordingly, condition (14) takes the form:

$$B \left(\zeta_1 - 1 + \frac{\omega^*}{\omega_0} \right) p_1 + Ap_1 \left(\frac{p_1}{\omega_0 \zeta_1 \dot{p}_1} \right)^{1/3} = 0 \tag{22}$$

Then the solution of Eq. (22) is as follows:

$$p_1 = c \exp \left\{ \left(\frac{A}{B(\omega^* - \omega_0)} \right)^3 \left[\ln \left(\frac{\omega_0 \zeta_1}{\omega^* - \omega_0 + \omega_0 \zeta_1} \right) + \right. \right.$$
$$\left. \left. + \frac{\omega^* - \omega_0}{\omega^* - \omega_0 + \omega_0 \zeta_1} + \frac{1}{2} \left(\frac{\omega^* - \omega_0}{\omega^* - \omega_0 + \omega_0 \zeta_1} \right)^2 \right] \right\} \tag{23}$$

As $\zeta_1 \to 0$, for finite values of the parameter $\omega^* - \omega_0$ we obtain a power-law eigenfunction of the form $p = c\zeta_1^\alpha$, which is in agreement with the results of V. Ya. Neiland [1974] and formula (4). The passage to the limit $\zeta \to 0$ and $\omega_0 \to \omega^*$ leads to function (21). Thus, in the subcritical flow regime continuous transition of the coordinate expansions near the leading edge to the expansions describing the beginning of the interaction in the flow containing the regions of both subcritical and supercritical flows is ensured.

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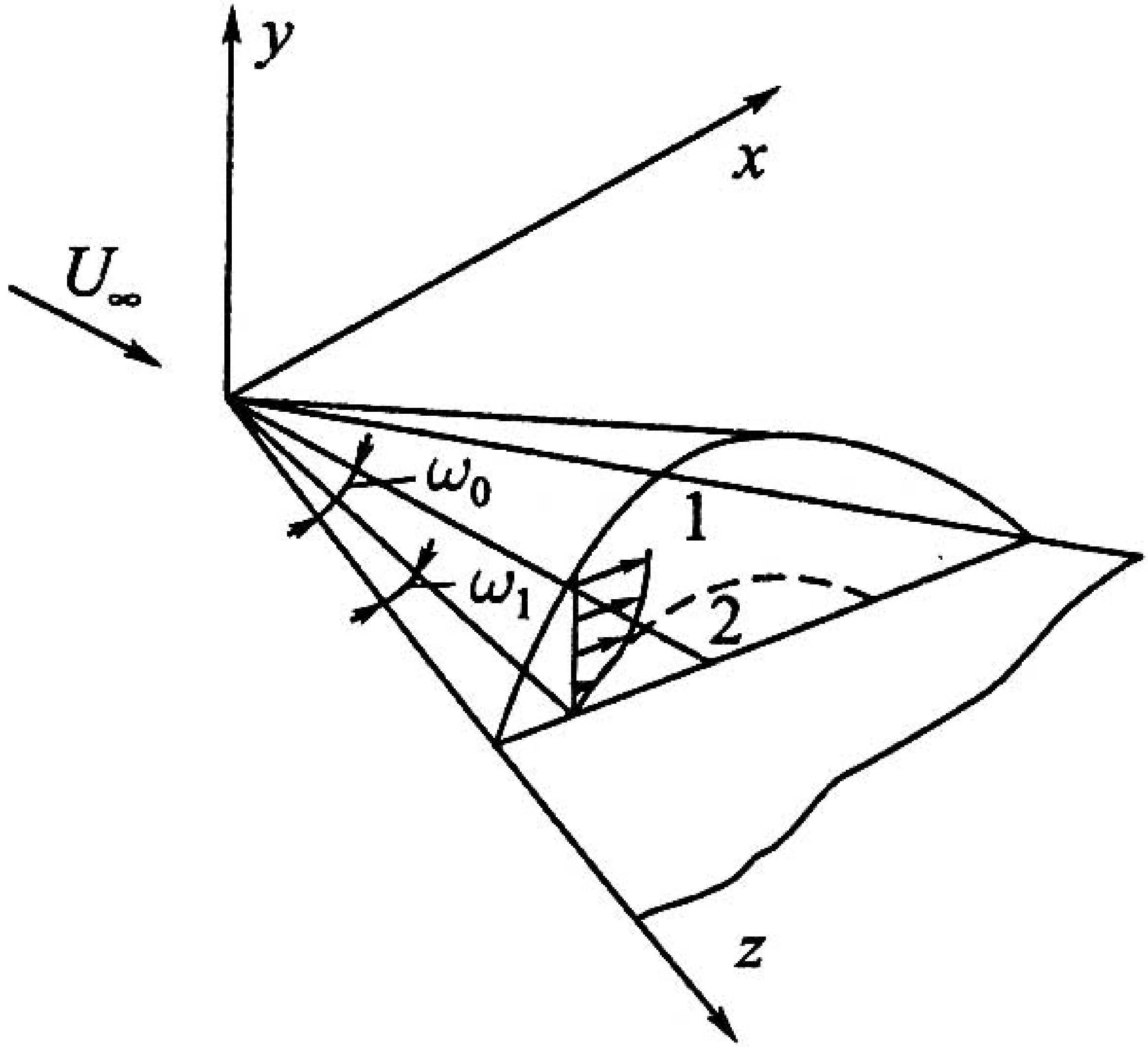
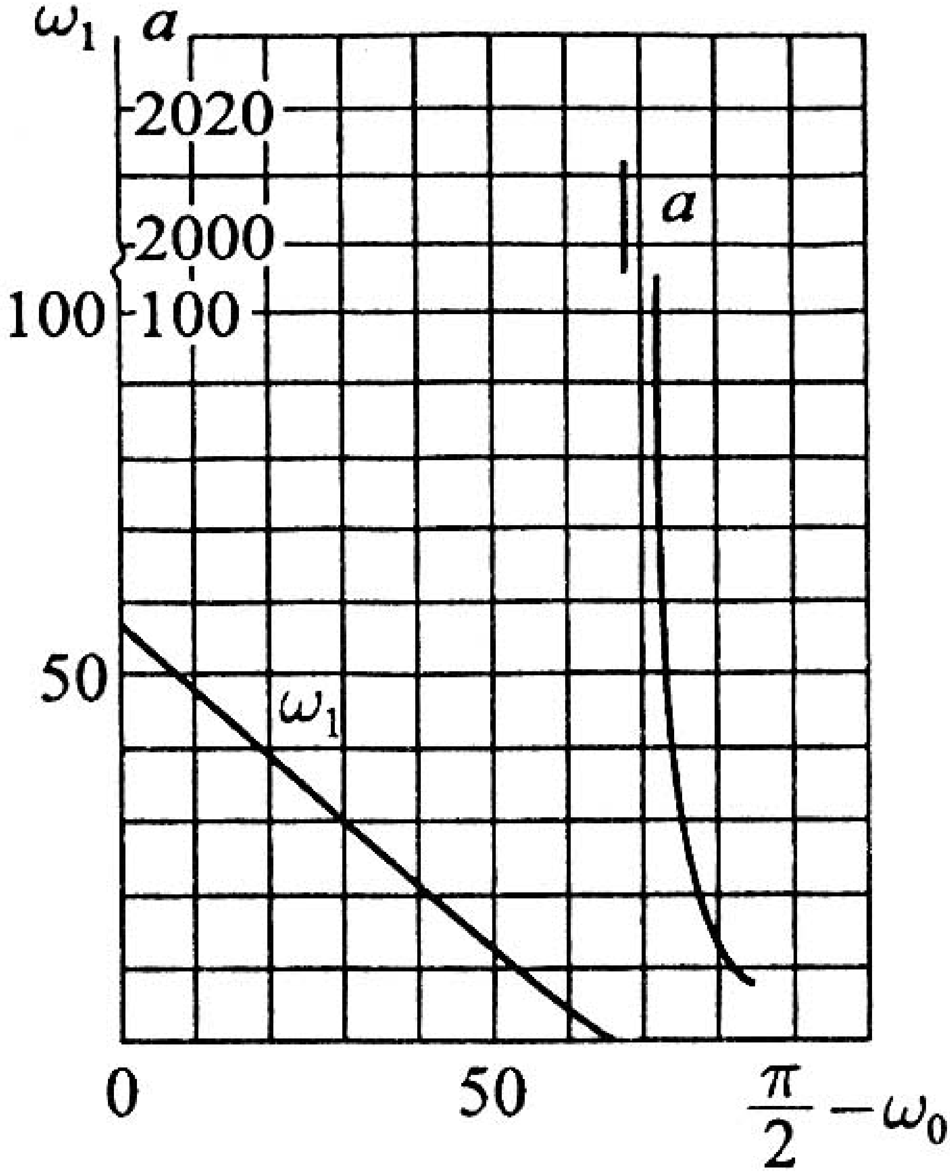


Figure 1:



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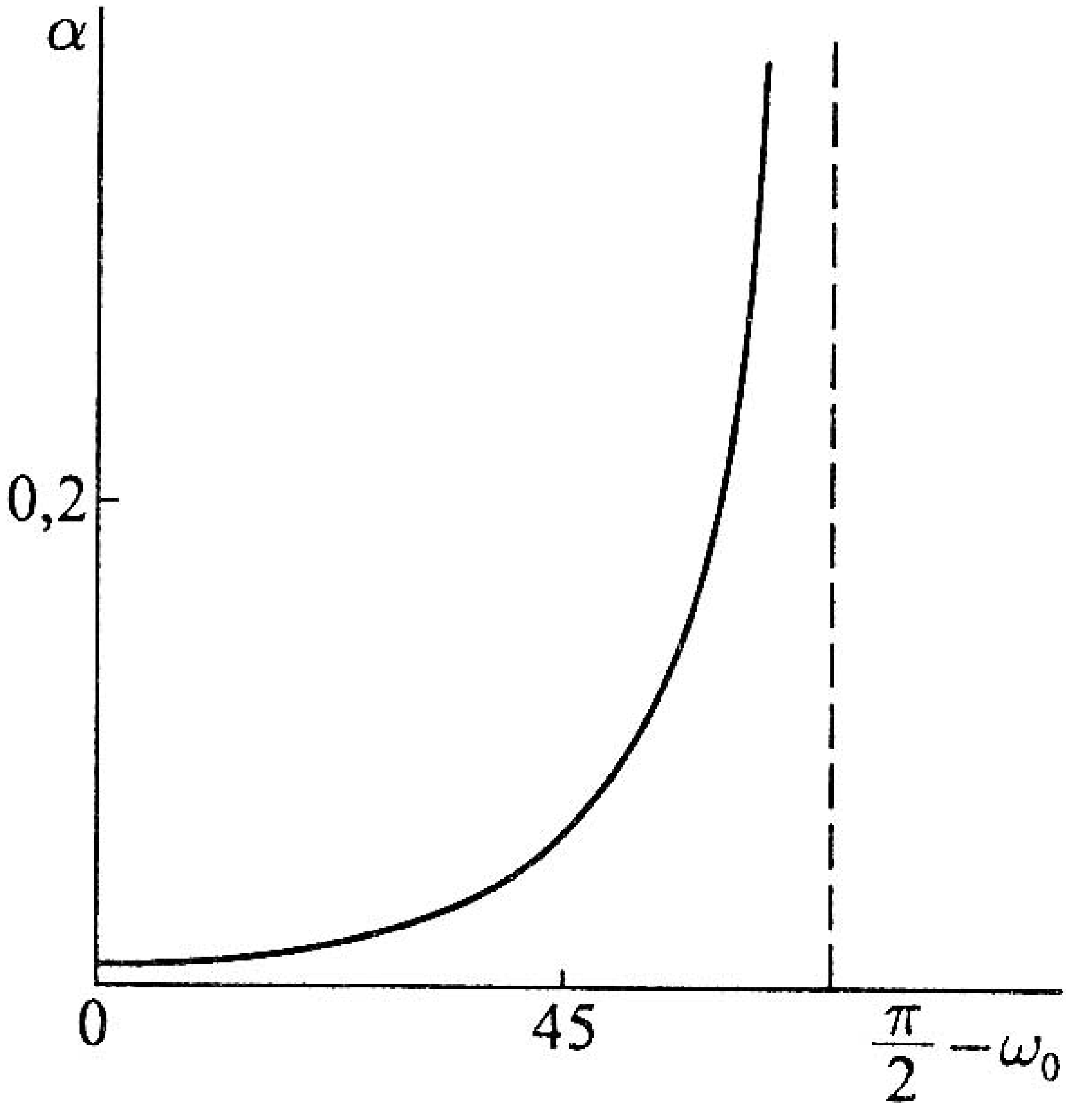

Figure 3:

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